On Higher Regularity of Discrete Hardy–Littlewood Maximal Functions

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Introduction

The **Hardy–Littlewood maximal function**, introduced in the 1930s, is a fundamental operator in harmonic analysis and PDEs. It provides a pointwise majorant of a function based on its local averages, making it essential for studying convergence and regularity.

Intuitively, $\mathcal{M}f(x)$ captures the *largest average value* of $|f|$ around x, measuring how "locally concentrated" a function can be. It bridges **local behavior** and **global estimates**, playing a central role in differentiation theorems and regularity problems.

Continuous Setting ($\mathbb{R}^d$)

The **centered** maximal function $\mathcal{M}$ is defined as the supremum of averages over balls $B(x,r)$ centered at x:

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy.$$

The **uncentered** maximal function $\widetilde{\mathcal{M}}$ takes the supremum over all balls containing x:

$$\widetilde{\mathcal{M}}f(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|\, dy.$$

Key property: $\widetilde{\mathcal{M}}f(x) \le C_d \mathcal{M}f(x)$, so both operators are equivalent up to constants, though the uncentered version is sometimes easier to handle.

Discrete Setting ($\mathbb{Z}$)

In the discrete setting, integrals are replaced by sums. The **centered discrete** maximal function $\mathbf{M}^d$ averages over symmetric intervals:

$$\mathbf{M}^d f(n) = \sup_{r\ge 0} \frac{1}{2r+1} \sum_{j=-r}^{r} |f(n+j)|.$$

The **uncentered discrete** maximal function $\widetilde{\mathbf{M}}^d$ averages over general intervals containing n:

$$\widetilde{\mathbf{M}}^d f(n) = \sup_{s,r\ge 0} \frac{1}{r+s+1} \sum_{j=-s}^{r} |f(n+j)|.$$

Applications and Intuition

Maximal functions are not only theoretical constructs: they appear naturally in

- studying pointwise convergence of Fourier series,
- differentiating integrals (**Lebesgue differentiation theorem**),
- regularity questions for solutions to PDEs such as the heat or Laplace equation,
- discrete analogues in number theory and combinatorics.

This dual role — connecting local averages to global estimates — makes maximal functions a key tool in modern analysis.

Literature Review: Boundedness Results

Many quantities of interest are dominated by the maximal function $\mathcal{M}f$. This raises the natural question:

How large can the maximal function of a given function be?

While pointwise control is not always possible, L^p-norm bounds are available.

L^p-Boundedness

- **G. H. Hardy and J. E. Littlewood** (1930) established the foundational results in one dimension ($\mathbb{R}$). For $1 < p \le \infty$, the maximal operator is bounded on $L^p(\mathbb{R})$, i.e., there exists $C_p > 0$ such that

  $$\|\mathcal{M}f\|_{L^p(\mathbb{R})} \le C_p \|f\|_{L^p(\mathbb{R})}.$$

  For $p = 1$, $\mathcal{M}$ is not bounded on $L^1(\mathbb{R})$, but satisfies the **weak-type (1,1)** inequality:

  $$|\{x \in \mathbb{R} : \mathcal{M}f(x) > \lambda\}| \le \frac{C}{\lambda}\|f\|_{L^1(\mathbb{R})}.$$

  Hence, $\mathcal{M}$ maps $L^1(\mathbb{R})$ into the weak Lebesgue space $L^{1,\infty}(\mathbb{R})$.

- These results were generalized to higher dimensions ($\mathbb{R}^n$) by **Norbert Wiener** (1939) using a Vitali-type covering lemma.

ℓ^p-Boundedness

Analogous results hold in the discrete setting. The discrete maximal operator $\mathbf{M}^d$ on $\mathbb{Z}^n$ satisfies the **weak-type (1,1)** bound:

$$|\{k \in \mathbb{Z}^n : \mathbf{M}^d f(k) > \lambda\}| \le \frac{C}{\lambda}\|f\|_{\ell^1(\mathbb{Z}^n)},$$

and the **strong-type (p,p)** bound for $1 < p \le \infty$:

$$\|\mathbf{M}^d f\|_{\ell^p(\mathbb{Z}^n)} \le C_p \|f\|_{\ell^p(\mathbb{Z}^n)}.$$

This theory was further developed by **J. Bourgain** (1980s–1990s), who obtained deep results for discrete averages over polynomial sequences and arithmetic progressions, especially in ergodic theory.

Motivation: A Competition of Effects

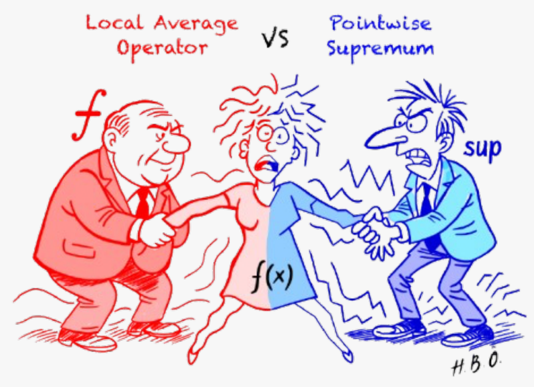
The regularity theory for $\mathcal{M}$ addresses a fundamental question:

If f is smooth (e.g., $f \in W^{1,p}$ or BV), is $\mathcal{M}f$ also smooth?

Two opposing mechanisms compete:

- **Averaging smooths:** $A_r f(x) = \frac{1}{|B_r|}\int_{B_r} f$ typically regularizes f.
- **Supremum roughens:** $\mathcal{M}f(x) = \sup_r A_r f(x)$ can introduce discontinuities and destroy smoothness.

Does the smoothing effect of averaging survive the roughening from the supremum?



Literature Review: Regularity Results

1. Continuous Setting

- **Kinnunen (1997)**: The centered maximal operator is bounded on Sobolev spaces,

  $$\mathcal{M} : W^{1,p}(\mathbb{R}^n) \longrightarrow W^{1,p}(\mathbb{R}^n),$$

  by the key pointwise inequality (a.e.)

  $$|\partial_i \mathcal{M}f(x)| \le \mathcal{M}(|\partial_i f|)(x).$$

  Keywords: *n-D, Centered, $W^{1,p}$*

The endpoint case $p = 1$ shows finer distinctions between centered and uncentered operators:

- **Tanaka (2002)**: For $f \in W^{1,1}(\mathbb{R})$, the function $\widetilde{\mathcal{M}}f$ is weakly differentiable and satisfies

  $$\|(\widetilde{\mathcal{M}}f)'\|_{L^1(\mathbb{R})} \le 2\,\|f'\|_{L^1(\mathbb{R})}.$$

  Keywords: *1-D, Uncentered, $W^{1,1}$*

- **Aldaz & Pérez Lázaro (2007)**: For $f \in BV(\mathbb{R})$, the uncentered maximal operator is absolutely continuous and satisfies

  $$\mathrm{Var}(\widetilde{\mathcal{M}}f) \le \mathrm{Var}(f).$$

  Keywords: *1-D, Uncentered, BV*

- **Kurka (2015)**: The centered operator preserves bounded variation up to a universal constant,

  $$\mathrm{Var}(\mathcal{M}f) \le 240000\ \mathrm{Var}(f).$$

  Keywords: *1-D, Centered, BV*

2. Discrete Setting

The discrete analogues parallel the continuous results:

- **Bober, Carneiro, Hughes & Pierce (2012)**: The discrete uncentered operator satisfies the contraction property

  $$\mathrm{Var}(\widetilde{\mathbf{M}}^d f) \le \mathrm{Var}(f).$$

  Keywords: *1-D, Uncentered, Discrete, BV*

- **Temur (2013)**: The discrete centered operator is bounded in variation,

  $$\mathrm{Var}(\mathbf{M}^d f) \le 294912004\ \mathrm{Var}(f).$$

  Keywords: *1-D, Centered, Discrete, BV*

Higher Order Regularity

Research on the maximal operator has mainly focused on first-order regularity until Temur (2022):

- **Temur (2022):** For any finite $A \subset \mathbb{Z}$ and $1 \le p \le \infty$,

  $$\|(\widetilde{\mathcal{M}^d}\chi_A)''\|_{\ell^p} \le 2^{1-1/p}3^{1/p}\|\chi_A''\|_{\ell^p}.$$

This result naturally raises new questions:

Q1: Extendable to broader functions?

- **Weigt (2024)** gave a counterexample.

Q2: Extendable to k-th order ($k \ge 2$)?

- **Temur, Ö. (2025)** generalized for any $k \ge 2$.

Reduction to Growth of $\|\chi_A^{(k)}\|_{\ell^1(\mathbb{Z})}$

Using the recurrence relation in discrete differenatiation

$$f^{(k)}(n) = f^{(k-1)}(n+1) - f^{(k-1)}(n).$$

we obtain

$$\|f^{(k)}\|_{\ell^p(\mathbb{Z})} \le 2\,\|f^{(k-1)}\|_{\ell^p(\mathbb{Z})}.$$

Applying recursively for discrete matimal operators and then using the first order regularity we obtain

$$\|\left(\widetilde{\mathbf{M}}^d\chi_A\right)^{(k)}\|_{\ell^p(\mathbb{Z})} \le 2^{k-1}\|\left(\widetilde{\mathbf{M}}^d\chi_A\right)'\|_{\ell^p(\mathbb{Z})} \lesssim_p \|\chi_A'\|_{\ell^p(\mathbb{Z})}.$$

Hence, to complete the k-th order regularity, we aim to prove:

$$\boxed{\|\chi_A'\|_{\ell^p(\mathbb{Z})} \lesssim_{k,p} \|\chi_A^{(k)}\|_{\ell^p(\mathbb{Z})},\ k \ge 2.}$$

Main Results on ℓ^p Growth of $\chi_A^{(k)}$

Theorem 1 (Temur, Ö. 2025): For any finite $A \subset \mathbb{Z}$ and $1 \le p \le \infty$,

$$\|\chi_A^{(k)}\|_{\ell^p(\mathbb{Z})} \ge (2k+1)^{-1/p}\|\chi_A'\|_{\ell^p(\mathbb{Z})}.$$

Keywords: *Exponential, Uniform Bound*

Theorem 2 (Temur, Ö. 2025): For any finite $A \subset \mathbb{Z}$, $n = \lfloor k/3 \rfloor$, and $1 \le p \le \infty$,

$$\|\chi_A^{(k)}\|_{\ell^p(\mathbb{Z})} \ge \tfrac{1}{3}\binom{k}{n}.$$

Keywords: *Exponential, Nonuniform Bound*

Theorem 3 (Temur, Ö. 2025): For any finite $A \subset \mathbb{Z}$, $1 \le p \le \infty$, and k sufficiently large depending on $|A|$,

$$\|\chi_A^{(k)}\|_{\ell^p(\mathbb{Z})} \ge 2^{k-1-\frac{|A|-1}{2}\log_2 2k}\left(\frac{\sqrt{|A|-1}}{7\pi e^{3/2}}\right)^{|A|-1}(k+1)^{-1/p'}|A|^{1/p}.$$

Keywords: *Asymptotic Bound via Nazarov–Turán*

Theorem 4 (Temur, Ö. 2025): For any finite $A \subset \mathbb{Z}$, $1 \le p \le \infty$, and k sufficiently large depending on $|A|$,

$$\|\chi_A^{(k)}\|_{\ell^p(\mathbb{Z})} \ge 2^{k-1-3(\log_2 e)(\frac{c\pi}{4})^{2/3}k^{1/3}}\left[(k+1)|A|\right]^{-1/p'}.$$

Keywords: *Asymptotic Bound via Borwein–Erdélyi*

Theorem 5 (Temur, Ö. 2025): For any finite $A \subset \mathbb{Z}$, $1 \le p \le \infty$, k sufficiently large depending on $|A|$, and any integer $0 \le a \le \log|A|$ with $\widehat{\chi_A}^{(a)}(1/2) \ne 0$,

$$\|\chi_A^{(k)}\|_{\ell^p(\mathbb{Z})} \ge 2^{k-\frac{a}{2}\log_2 2k+a-2}\left(\frac{a}{e}\right)^{a/2}\frac{1}{a!}\left[(k+1)|A|\right]^{-1/p'}.$$

Keywords: *Asymptotic Bound via Borwein, Erdélyi, Kós*

Theorem 6 (Temur, Ö. 2025): If $A \subset \mathbb{Z}$ satisfies

$$\sum_{m<n\in A}\frac{1}{n-m} \le \frac{\pi}{8}|A|,$$

then for any $k \in \mathbb{N}$ and $1 \le p \le \infty$,

$$\|\chi_A^{(k)}\|_{\ell^p(\mathbb{Z})} \ge \begin{cases} (k+1)^{1/p-1/2}2^{k/2-1}\|\chi_A\|_{\ell^p(\mathbb{Z})}, & p \ge 2,\\ 2^{k-(k+2)/p}\|\chi_A\|_{\ell^p(\mathbb{Z})}, & 1 \le p < 2.\end{cases}$$

Keywords: *Exponential Growth, Sparsity Condition*

Sketch Proof of Theorem 3 via Nazarov–Turán

Fourier-analytic Framework

1. **Discrete Fourier Transform:** Let $\mathbb{T} = \mathbb{R}/\mathbb{Z} \simeq [0,1)$ and $f : \mathbb{Z} \to \mathbb{C}$ finitely supported:

   $$\widehat{f}(x) = \sum_{n\in\mathbb{Z}} f(n)e^{-2\pi i n x}, \quad x \in \mathbb{T}.$$

   For $k \ge 1$:

   $$\widehat{f^{(k)}}(x) = (e^{2\pi i x}-1)^k\widehat{f}(x),$$

   so the discrete derivative is diagonalized by the Fourier transform.

2. **ℓ^p Estimates via Hausdorff–Young:** For $1 \le p \le 2$ with conjugate p':

   $$\|\widehat{g}\|_{L^{p'}(\mathbb{T})} \le \|g\|_{\ell^p(\mathbb{Z})}.$$

   Apply to $g = \chi_A^{(k)}$:

   $$\|(e^{2\pi i x}-1)^k\widehat{\chi_A}(x)\|_{L^{p'}(\mathbb{T})} \le \|\chi_A^{(k)}\|_{\ell^p(\mathbb{Z})}.$$

   In particular, for $p = 1$:

   $$\|\chi_A^{(k)}\|_{\ell^1(\mathbb{Z})} \ge \sup_{x\in\mathbb{T}}|(e^{2\pi i x}-1)^k\widehat{\chi_A}(x)|.$$

3. **Localization near $x = 1/2$:** $m(x) = e^{2\pi i x} - 1 = 2i\sin(\pi x)e^{\pi i x}$, so $|m(x)| = 2|\sin(\pi x)|$. Maximum at $x = 1/2$, restrict to $E_r = (1/2-r, 1/2+r)$:

   $$|m(x)| \ge 2 - \pi^2 r^2, \qquad \|\chi_A^{(k)}\|_{\ell^1(\mathbb{Z})} \ge (2-\pi^2 r^2)^k\|\widehat{\chi_A}\|_{L^\infty(E_r)}.$$

4. **Nazarov–Turán Inequality:** For $P(x) = \sum_{n\in A} a_n e^{2\pi i n x}$ and measurable $E \subset \mathbb{T}$:

   $$\|P\|_{L^\infty(\mathbb{T})} \le \left(\frac{14e}{|E|}\right)^{|A|-1}\|P\|_{L^\infty(E)}.$$

   Apply to $P(x) = \widehat{\chi_A}(-x)$, $|E_r| = 2r$:

   $$\|\widehat{\chi_A}\|_{L^\infty(E_r)} \ge \left(\frac{r}{7e}\right)^{|A|-1}|A|,$$

   hence

   $$\|\chi_A^{(k)}\|_{\ell^1(\mathbb{Z})} \ge (2-\pi^2r^2)^k\left(\frac{r}{7e}\right)^{|A|-1}|A|.$$

5. **Optimization and Extension to ℓ^p:** Optimizing r and applying Hölder's inequality extends the bound to general ℓ^p:

   $$\|\chi_A^{(k)}\|_{\ell^p(\mathbb{Z})} \ge 2^{k-1-\frac{|A|-1}{2}\log_2 2k}\left(\frac{\sqrt{|A|-1}}{7\pi e^{3/2}}\right)^{|A|-1}(k+1)^{-1/p'}|A|^{1/p}.$$

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