Hardy-Littlewood Maximal Function

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- **Lebesgue Measure**

  The Lebesgue measure of a ball $B(x,r) \subset \mathbb{R}^n$ centered at x with radius r is denoted by $|B(x,r)|$ and represents its volume.

  For instance, the Lebesgue measure of $B(x,r) \subset \mathbb{R}^3$ is

$$|B(x,r)| = \frac{4}{3}\pi r^3.$$

- **Average Function**

  For $f \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}^d)$ and a ball $B(x,r)$, the average of f over $B(x,r)$ is defined as

$$A_r f(x) = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y)\, dy.$$

- **Lebesgue Spaces ($\mathcal{L}^p$ spaces)**

  - $1 \leq p < \infty$:

$$\mathcal{L}^p(\mathbb{R}^d) = \left\{ f : \mathbb{R}^d \to \mathbb{R} \,\middle|\, \int_{\mathbb{R}^d} |f(x)|^p \, dx < \infty \right\},$$

and the $\mathcal{L}^p$-norm is given by

$$\|f\|_{\mathcal{L}^p} = \left(\int_{\mathbb{R}^d} |f(x)|^p \, dx \right)^{1/p}$$

  - $p = \infty$:

$$\mathcal{L}^\infty(\mathbb{R}^d) = \left\{ f : \mathbb{R}^d \to \mathbb{R} \,\middle|\, \sup_{x \in \mathbb{R}^d} |f(x)| < \infty \right\}.$$

The $\mathcal{L}^\infty$-norm is given by

$$\|f\|_{\mathcal{L}^\infty} = \sup_{x \in \mathbb{R}^d} |f(x)|.$$

- **Sobolev Spaces $\mathcal{W}^{k,p}$**

  For $1 \leq p \leq \infty$, we define

  $$\mathcal{W}^{1,p}(\mathbb{R}^d) = \left\{ f \in \mathcal{L}^p(\mathbb{R}^d) \;\middle|\; D_i f \in \mathcal{L}^p(\mathbb{R}^d) \text{ for all } i = 1, 2, \cdots, d \right\},$$

  where D_i denotes the weak partial derivative of f in the direction e_i.

  The $\mathcal{W}^{1,p}$-norm of is given by

  $$\|f\|_{\mathcal{W}^{1,p}} = \|f\|_{\mathcal{L}^p} + \left(\sum_{i=1}^{d} \|D_i f\|_{\mathcal{L}^p} \right).$$

- **Strong Type** (p, p)

  An operator T is said to be of *strong type* (p, p) for $1 \leq p \leq \infty$ if there exists a constant $C > 0$ such that for all functions f in the domain of T,

  $$\|Tf\|_p \leq C\|f\|_p,$$

  where $\|\cdot\|_p$ denotes the L^p-norm.

- **Weak Type** (p, p)

  An operator T is said to be of *weak type* (p, p) for $1 \leq p < \infty$ if there exists a constant $C > 0$ such that for all functions f and for all $\lambda > 0$,

  $$|\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}| \leq \frac{C\|f\|_p^p,}{\lambda^p}.$$

Classical Hardy-Littlewood Maximal Function

- The **centered Hardy-Littlewood maximal function** $\mathcal{M}$ of a locally integrable function f on $\mathbb{R}^d$ is defined as

$$\mathcal{M}f(x) := \sup_{r>0} \underbrace{\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy}_{A_r|f|(x)}$$

- The **non-centered Hardy-Littlewood maximal function** $\tilde{\mathcal{M}}$ of a locally integrable function f on $\mathbb{R}^d$ is defined as

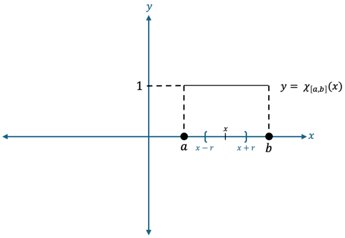
$$\tilde{\mathcal{M}}f(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|\, dy,$$

where the supremum is taken over all balls B in $\mathbb{R}^d$ that contain the point x.

Compute $\mathcal{M}\chi_{[a,b]}$

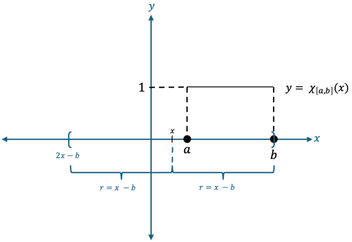
Case 1: If $x \in (a,b)$, then we can choose r such that $(x-r, x+r) \subset [a,b]$.

$$\begin{aligned}
\mathcal{M}\chi_{[a,b]}(x) &= \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} \chi_{[a,b]}(y)\, dy \\
&= \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} 1\, dy \\
&= \sup_{r>0} 1 \\
&= 1.
\end{aligned}$$



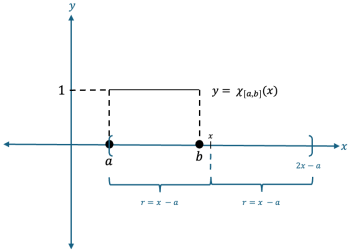
Case 2: If $x \leq a$, then the value of r that gives the maximal averages is $r = b - x$. Thus,

$$\begin{aligned}\mathcal{M}\chi_{[a,b]}(x) &= \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} \chi_{[a,b]}(y)\, dy \\ &= \frac{1}{2(b-x)} \int_{2x-b}^{b} \chi_{[a,b]}(y)\, dy \\ &= \frac{b-a}{2(b-x)}.\end{aligned}$$



Case 3: Similarly, if $x \geq b$, the value of r that gives the supremum above is $r = x - a$.

$$\begin{aligned}\mathcal{M}\chi_{[a,b]}(x) &= \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} \chi_{[a,b]}(y)\, dy \\ &= \frac{1}{2(x-a)} \int_{a}^{2x-a} \chi_{[a,b]}(y)\, dy \\ &= \frac{b-a}{2(x-a)}.\end{aligned}$$



As a result, we have

$$\mathcal{M}\chi_{[a,b]}(x) = \begin{cases} 1 & \text{if } x \in (a,b) \\ \frac{b-a}{2(b-x)} & \text{if } x \leq a \\ \frac{b-a}{2(x-a)} & \text{if } x \geq b \end{cases}$$

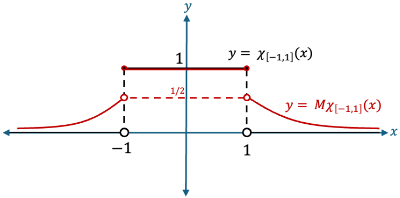


Figure: $\chi_{[-1,1]}$ VS $\mathcal{M}\chi_{[-1,1]}$

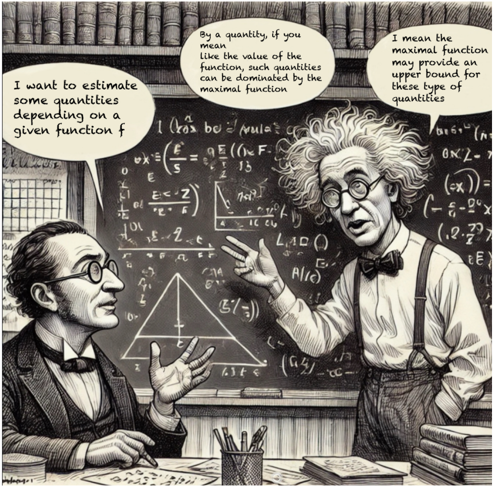


Figure: An Idea to Estimate Some Quantities Depending on Function

Question: How "big" can the maximal function $\mathcal{M}f$ of a given function f be?

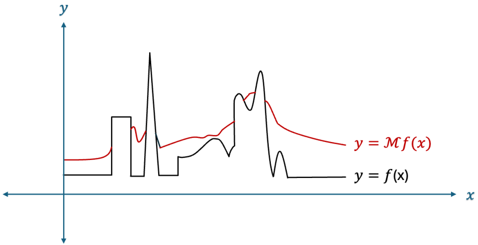


Figure: Pointwise inequalities are not always possible.

! But, the bounds in the $\mathcal{L}^p$-sense can be given.

(i) **Boundedness on $\mathcal{L}^p(\mathbb{R}^d)$:** For any $1 < p \leq \infty$, there exists a constant $C_p > 0$ such that

$$\|\mathcal{M}f\|_{\mathcal{L}^p(\mathbb{R}^d)} \leq C_p \|f\|_{\mathcal{L}^p(\mathbb{R}^d)}.$$

This means that

$$\mathcal{M} : \mathcal{L}^p(\mathbb{R}^d) \longrightarrow \mathcal{L}^p(\mathbb{R}^d)$$

is bounded.

In other words, the Hardy-Littlewood maximal function is of strong type (p, p), where $1 < p \leq \infty$.

! But, when $p = 1$, this result fails.

- **Counterexample:** Consider the maximal function $\mathcal{M}\chi_{[0,1]}(x)$ of the characteristic function $\chi_{[0,1]}(x) \in \mathcal{L}^1(\mathbb{R})$:

$$\begin{aligned}\mathcal{M}\chi_{[0,1]}(x) &= \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} \chi_{[0,1]}(y)\, dy \\ &\geq \frac{1}{2x} \int_0^{2x} \chi_{[0,1]}(y)\, dy = \frac{1}{2x} \int_0^1 1\, dy = \frac{1}{2x} \notin \mathcal{L}^1(\mathbb{R}),\end{aligned}$$

provided that $x \geq 1$. Hence, $\mathcal{M}\chi_{[0,1]} \notin \mathcal{L}^1(\mathbb{R})$.

(ii) **Weak-type** $(1,1)$ **Inequality**: There exists a constant C_1 such that for any $\lambda > 0$,
$$|\{x \in \mathbb{R}^d : \mathcal{M}f(x) > \lambda\}| \leq \frac{C_1}{\lambda} \cdot \|f\|_{\mathcal{L}^1(\mathbb{R}^d)}.$$

This indicates that the Hardy-Littlewood maximal function is a bounded operator from $\mathcal{L}^1(\mathbb{R}^d)$ to the weak Lebesgue space $\mathcal{L}^1_w(\mathbb{R}^d)$.

Define the set $E_\lambda = \{x \in \mathbb{R}^n : (Mf)(x) > \lambda\}$. We want to show that

$$|E_\lambda| \leq \frac{C}{\lambda}\|f\|_{\mathcal{L}^1},$$

where C is a constant.

Step1: Selection of Balls

- For each $x \in E_\lambda$, there exists a ball $B(x, r_x)$ such that

$$\frac{1}{|B(x,r_x)|}\int_{B(x,r_x)} |f(y)|\, dy > \lambda.$$

  Let $\mathcal{B} = \{B(x,r_x) \ : \ x \in E_\lambda\}$.

Sketch of the Proof of the Weak-type $(1, 1)$ Inequality

Step2: Vitali Covering Lemma

- Apply the Vitali Covering Lemma to extract a disjoint subcollection $\{B(x_i, r_i)\}_{i \in I} \subseteq \mathcal{B}$ such that

$$E_\lambda \subseteq \bigcup_{i \in I} B(x_i, 3r_i).$$

Step3: Estimate the Measure

- The measure of the set E_λ is bounded by the measure of the union of the balls with radius $3r_i$:

$$|E_\lambda| \leq \sum_{i \in I} |B(x_i, 3r_i)| = 3^d \sum_{i \in I} |B(x_i, r_i)|.$$

Sketch of the Proof of the Weak-type $(1,1)$ Inequality

For each $i \in I$, we have

$$\lambda < \frac{1}{|B(x_i, r_i)|} \int_{B(x_i,r_i)} |f(y)|\, dy \implies \lambda |B(x_i, r_i)| < \int_{B(x_i,r_i)} |f(y)|\, dy.$$

Summing these inequalities over the disjoint balls, we obtain

$$\lambda \sum_{i \in I} |B(x_i, r_i)| < \sum_{i \in I} \int_{B(x_i,r_i)} |f(y)|\, dy \le \int_{\mathbb{R}^n} |f(y)|\, dy = \|f\|_1.$$

Thus,

$$|E_\lambda| \le 3^d \sum_{i \in I} |B(x_i, r_i)| \le 3^d \frac{\|f\|_1}{\lambda},$$

where $C = 3^d$. $\square$

Marcinkiewicz Interpolation Theorem:

Let T be a sublinear operator defined on $\mathbb{R}^d$. Suppose T is of weak type (p, p) and (q, q), where $1 \leq p < q \leq \infty$. Then T is of strong type (r, r) for all $r \in (p, q)$:

$$\|Tf\|_r \leq C_r \|f\|_r.$$

We have already known that

- $\mathcal{M}$ is sublinear,
- $\mathcal{M}$ is of weak type $(1, 1)$ and (∞, ∞).

By the Marcinkiewicz Interpolation Theorem, $\mathcal{M}$ is of strong type (p, p) for all $1 < p < \infty$. $\square$

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Main Goal of Regularity Theory

- Investigating the action of the Hardy-Littlewood maximal operator $\mathcal{M}$ on various function spaces such as Lebesgue Spaces $\mathcal{L}^p$, Sobolev Spaces $\mathcal{W}^{k,p}$, etc.
- This analysis helps determine whether $\mathcal{M}$ improves, preserves, or destroys the initial **regularity properties** of a function f, which are:
  - **Continuity:** Does $\mathcal{M}$ map continuous functions to continuous functions?
  - **Differentiability:** Does $\mathcal{M}$ preserve or destroy differentiability?
  - **Smoothness:** What is the impact on higher-order derivatives and smoothness classes?

In 1997, J. Kinnunen showed that the Hardy-Littlewood maximal operator $\mathcal{M}$ is bounded on Sobolev spaces $\mathcal{W}^{1,p}(\mathbb{R}^d)$ for $1 < p \leq \infty$.

In fact, he proved the pointwise inequality:

Thereom (J. Kinnunen, 1997)

Let $1 < p \leq \infty$. If $f \in \mathcal{W}^{1,p}(\mathbb{R}^d)$, then $\mathcal{M}f \in \mathcal{W}^{1,p}(\mathbb{R}^d)$ and

$$|D_i \mathcal{M} f| \leq \mathcal{M}(D_i f), \quad i = 1, \ldots, d,$$

almost everywhere in $\mathbb{R}^d$. Therefore, $\mathcal{M} : \mathcal{W}^{1,p}(\mathbb{R}^d) \longrightarrow \mathcal{W}^{1,p}(\mathbb{R}^d)$ is bounded.

J. Kinnunen, The Hardy-Littlewood maximal function of a Sobolev function, Israeli Journal of Mathematics, 100(1), 117-124, 1997.

Key Idea: Averaging is traditionally viewed as a smoothing process.

Central Question: Do smoothing properties persist when taking the pointwise supremum over averages?

Important Points: For any non-identically zero $f \in \mathcal{L}^1(\mathbb{R}^d)$,

- $\mathcal{M}f \notin \mathcal{L}^1(\mathbb{R}^d)$.
- But, the interesting challenge is controlling the behavior of the derivative of the maximal function.

Question: (Hajłasz & Onninen)

Is the operator

$$\begin{aligned} D_i\mathcal{M} : \mathcal{W}^{1,1}(\mathbb{R}^d) &\longrightarrow \mathcal{L}^1(\mathbb{R}^d) \\ f &\mapsto D_i\mathcal{M}f \end{aligned}$$

bounded?

Thereom (H. Tanaka, 2002)

If $f \in \mathcal{W}^{1,1}(\mathbb{R})$, then $\tilde{\mathcal{M}}f$ is weakly differentiable and

$$\|\tilde{\mathcal{M}}(f)'\|_{\mathcal{L}^1(\mathbb{R})} \leq 2\|f'\|_{\mathcal{L}^1(\mathbb{R})}$$

H. Tanaka, A remark on the derivative of the one-dimensional Hardy-Littlewood maximal function, Bull. Austral. Math. Soc., 65(2), 253-258, 2002.

Thereom (Aldaz and Pérez Lázaro, 2007)

Let $f : \mathbb{R} \to \mathbb{R}$ be a function of bounded variation. Then $\tilde{\mathcal{M}}f$ is an absolutely continuous function and we have the inequality

$$Var(\tilde{\mathcal{M}}f) \leq Var(f),$$

where $Var(f)$ denote the total variation f.

J. M. Aldaz and J. Pérez Lázaro, Functions of bounded variation, the derivative of the one dimensional maximal function, and applications to inequalities, Trans. Amer. Math. Soc., 259(5), 2243-2461, 2007.

Regularizing Effect:

- The non-centered maximal operator $\tilde{\mathcal{M}}$ transforms a function of bounded variation into an absolutely continuous function.
- The centered maximal operator $\mathcal{M}$ does not share this regularizing effect.
- The non-centered operator $\tilde{\mathcal{M}}$ is more regular than the centered operator $\mathcal{M}$.

Thereom (O. Kurka, 2015)

Let $f : \mathbb{R} \to \mathbb{R}$ be a function of bounded variation. Then

$$Var(\mathcal{M}f) \leq 240004\, Var(f).$$

O. Kurka, On the variation of the Hardy-Littlewood maximal function, and applications to inequalities, Ann. Acad. Sci. Fenn. Math., 40, 109-133, 2015.

Question: (Hajłasz & Onninen)

Is the operator

$$\begin{aligned} D_i\mathcal{M} : \mathcal{W}^{1,1}(\mathbb{R}^d) &\longrightarrow \mathcal{L}^1(\mathbb{R}^d) \\ f &\mapsto D_i\mathcal{M}f \end{aligned}$$

bounded?

! The question is still open for $d \geq 2$.

- For a function $f : \mathbb{Z} \to \mathbb{R}$, we define the **discrete centered Hardy-Littlewood maximal function M** as

$$\mathbf{M}f(n) := \sup_{r \in \mathbb{Z}^+} \frac{1}{2r+1} \sum_{j=-r}^{r} |f(n+j)|,$$

where the supremum is taken over nonnegative and integer values of r.

- The **non-centered discrete Hardy-Littlewood maximal function $\tilde{\mathbf{M}}$** is defined as

$$\tilde{\mathbf{M}}f(n) := \sup_{r,s \in \mathbb{Z}^+} \frac{1}{r+s+1} \sum_{j=-r}^{s} |f(n+j)|,$$

where the supremum is taken over nonnegative and integer values of r and s.

Convention in Discrete Setting

For $f : \mathbb{Z} \to \mathbb{R}$,

- ℓ^p-**norm** for $1 \leq p < \infty$:

$$\|f\|_{\ell^p(\mathbb{Z})} = \left(\sum_{n=-\infty}^{\infty} |f(n)|^p \right)^{1/p}$$

- ℓ^∞-**norm**:

$$\|f\|_{\ell^\infty(\mathbb{Z})} = \sup_{n \in \mathbb{Z}} |f(n)|$$

Consequently, we define

$$\ell^p(\mathbb{Z}) = \{f : \mathbb{Z} \to \mathbb{R} \quad \|f\|_{\ell^p(\mathbb{Z})} < \infty\},$$

where $1 \leq p \leq \infty$.

Convention in Discrete Setting

We define the derivatives of a discrete function by

$$\begin{aligned} f'(n) &= f(n+1) - f(n), \\ f''(n) &= f(n+2) - 2f(n+1) + f(n), \\ f'''(n) &= f(n+3) - 3f(n+2) + 3f(n+1) - f(n), \\ &\vdots \end{aligned}$$

Note that since $\|f^{(k)}\|_{\ell^p(\mathbb{Z})} \leq 2^k \|f\|_{\ell^p(\mathbb{Z})}$, the analogous Sobolev spaces $\mathcal{W}^{k,p}(\mathbb{Z})$ are again the spaces ℓ^p.

Thereom (Bober, Carneiro, Pierce, Hughes, 2012)

If $f : \mathbb{Z} \to \mathbb{R}$ be a function of bounded variation, then

$$Var(\tilde{\mathbf{M}} f) \leq Var(f).$$

J. Bober, E. Carneiro, K. Hughes and L. B. Pierce, On a discrete version of Tanaka's theorem for maximal functions, Proc. Amer. Math. Soc., 140, 1669-1680, 2012.

Thereom (Temur, 2013)

If $f : \mathbb{Z} \to \mathbb{R}$ be a function of bounded variation, then

$$Var(\mathbf{M}f) \leq C Var(f),$$

with $C = (72000)2^{12} + 4 = 294912004$.

F. Temur, On regularity of the discrete Hardy-Littlewood maximal function, preprint at http://arxiv.org/abs/1303.3993.

! As in the continuous case, the non-centered version is more friendly for the sort of questions we investigate here.

- Despite the vast effort expended on understanding the first derivatives of maximal functions, very little attention is paid to **higher order derivatives**.
- We obtain the first positive result for the discrete non-centered maximal functions by restricting the function class concerned.

Thereom (Temur, 2022)

Let $A \subseteq \mathbb{Z}$. Then for characteristic function χ_A,

$$\|(\tilde{\mathbf{M}}\chi_A)''\|_{\ell^1} \leq 3\|\chi_A''\|_{\ell^1}.$$

F. Temur, The second derivative of the discrete Hardy-Littlewood maximal function, preprint at https://arxiv.org/abs/2205.03953v1.

Directions of Extending

Three possible directions of extending this result emerge.

(1) Can you extend the result to to other ℓ^p-norms?

(2) Can you extend the result to higher order derivatives?

(3) Can you extend the result to more general classes of functions?

Thereom (Temur, Özcan, ∼2023)

Let $A \subseteq \mathbb{Z}$. Then for characteristic function χ_A,

$$\|(\tilde{\mathbf{M}}\chi_A)''\|_{\ell^p} \leq 2^{1/p} \cdot 3\|\chi_A''\|_{\ell^p}.$$

Recently, we have observed that

$$\|\tilde{\mathbf{M}}_{\chi_A}^{(k+1)}\|_{\ell_1} \leq 2 \cdot \|\tilde{\mathbf{M}}_{\chi_A}^{(k)}\|_{\ell_1},$$

for any $k \geq 2$. This leads to

$$\|\tilde{\mathbf{M}}_{\chi_A}^{(k)}\|_{\ell_1} \leq 2^{k-2} \cdot \|(\tilde{\mathbf{M}}_{\chi_A})''\|_{\ell_1} \leq 2^{k-2} \cdot 3 \cdot \|\chi_A''\|_{\ell_1}.$$

Question: *Does the inequality* $\|\chi_A''\|_{\ell_1(\mathbb{Z})} \leq C \cdot \|\chi_A^{(k)}\|_{\ell_1(\mathbb{Z})}$ *hold true for any* $k \geq 3$?

Fail to Extend to General Classes of Functions

Recently, we have demonstrated a very simple and almost explicit function f for which the second derivatives in the weak sense of $\mathcal{M}f$, $\mathbf{M}f$ do not even exist as functions.


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