On the Regularity of the Discrete Hardy–Littlewood Maximal Function

Hikmet Burak Özcan

İzmir Institute of Technology

I. Workshop on Applied Mathematics and Statistics
İzmir University of Economics

May 10, 2025

Table of Contents

Table of Contents

Let $f \in L^1_{\text{loc}}(\mathbb{R}^d)$. The **Hardy–Littlewood maximal function** of f, denoted $\mathcal{M}f$, is defined for each $x \in \mathbb{R}^d$ by

$$\mathcal{M}f(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy,$$

where $B(x,r)$ denotes the open ball of radius r centered at x, and $|\cdot|$ denotes Lebesgue measure in $\mathbb{R}^d$.

The **non-centered Hardy–Littlewood maximal function** is defined by

$$\widetilde{\mathcal{M}}f(x) := \sup_{\substack{B \subset \mathbb{R}^d \\ x \in B}} \frac{1}{|B|} \int_B |f(y)|\, dy,$$

where the supremum is taken over all balls $B \subset \mathbb{R}^d$ containing the point x.

Let $f : \mathbb{Z} \to \mathbb{R}$. The **discrete Hardy–Littlewood maximal function**, denoted $\mathbf{M}f$, is defined for each $n \in \mathbb{Z}$ by

$$\mathbf{M}f(n) := \sup_{r \in \mathbb{Z}_{\geq 0}} \frac{1}{2r+1} \sum_{j=-r}^{r} |f(n+j)|,$$

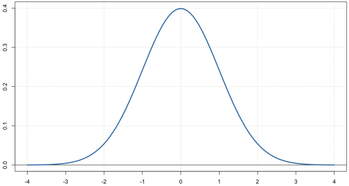
where the supremum is taken over all nonnegative integers r.

The **non-centered discrete Hardy–Littlewood maximal function**, denoted $\widetilde{\mathbf{M}}f$, is given by

$$\widetilde{\mathbf{M}}f(n) := \sup_{r,s \in \mathbb{Z}_{\geq 0}} \frac{1}{r+s+1} \sum_{j=-r}^{s} |f(n+j)|,$$

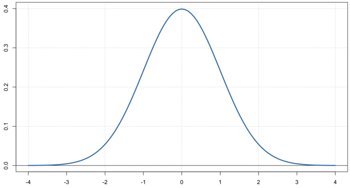
where the supremum is taken over all pairs of nonnegative integers r and s.

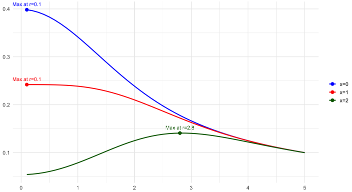
Example: Hardy-Littlewood Maximal Function of the Standard Normal Distribution



$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Example: Hardy-Littlewood Maximal Function of the Standard Normal Distribution

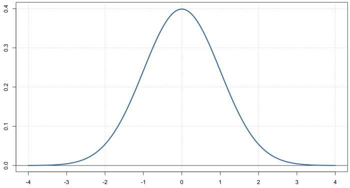


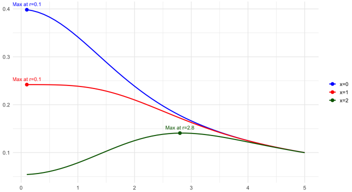


$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$A_r|f|(x) = \frac{1}{2r} \int_{x-r}^{x+r} |f(t)| dt$$

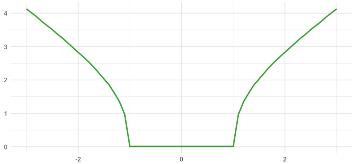
Example: Hardy-Littlewood Maximal Function of the Standard Normal Distribution





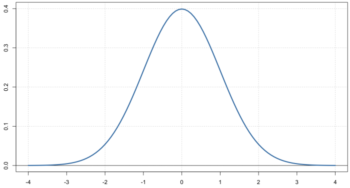
$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

$$A_r|f|(x) = \frac{1}{2r} \int_{x-r}^{x+r} |f(t)| dt$$

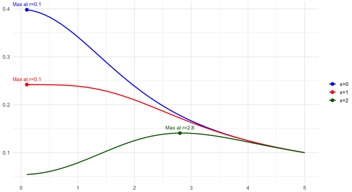


Optimal Radius Values for $A_r|f|(x)$

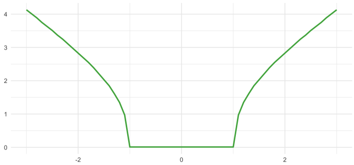
Example: Hardy-Littlewood Maximal Function of the Standard Normal Distribution



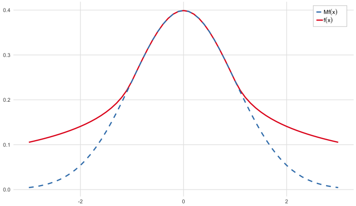
$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$



$$A_r|f|(x) = \frac{1}{2r}\int_{x-r}^{x+r} |f(t)|dt$$

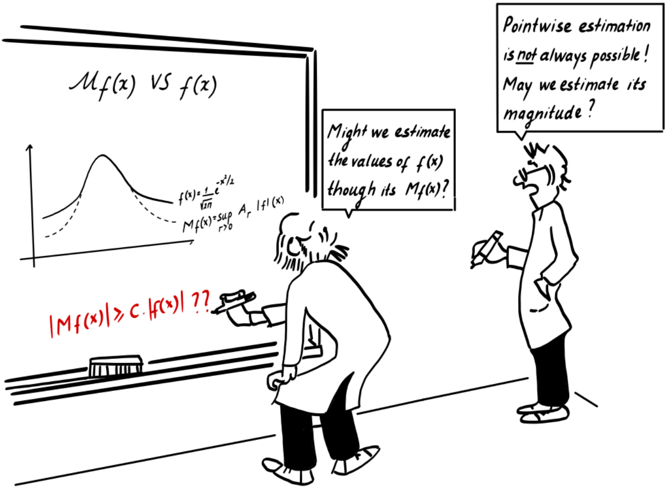


Optimal Radius Values for $A_r|f|(x)$



$\mathcal{M}f(x)$ **VS** $f(x)$

How "big" can the maximal function $\mathcal{M}f$ of a given function f be?



Theorem (G. H. Hardy and J. E. Littlewood, 1930)

Let $d \in \mathbb{N}$. The Hardy–Littlewood maximal operator

$$\mathcal{M} \colon L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d), \quad \mathcal{M}f(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)|\, dy,$$

satisfies the following key properties:

1. **Strong Boundedness ($L^p \to L^p$)**
   For every $1 < p \leq \infty$, $\mathcal{M}$ is a bounded operator–that is, there exists $C_p > 0$ such that
   $$\|\mathcal{M}f\|_{L^p} \leq C_p \|f\|_{L^p} \quad \forall f \in L^p(\mathbb{R}^d).$$
2. **Weak Boundedness ($L^1 \to L^{1,\infty}$)**
   On $L^1(\mathbb{R}^d)$, $\mathcal{M}$ is weakly bounded–that is, there exists $C_1 > 0$ such that for all $\lambda > 0$,
   $$\left|\{x \in \mathbb{R}^d : \mathcal{M}f(x) > \lambda\}\right| \leq \frac{C_1}{\lambda} \|f\|_{L^1}.$$
   This means $\mathcal{M}$ maps L^1 into the weak L^1 space.

Table of Contents

Guiding Question: Given a function f belonging to a regularity class $\mathcal{C}$, what can be said about the regularity of $\mathcal{M}f$?

For instance:

- **Does the maximal operator $\mathcal{M}$ preserve differentiability properties of f?**
- **If $f \in W^{1,p}(\mathbb{R}^d)$, is it true that $\mathcal{M}f \in W^{1,p}(\mathbb{R}^d)$?**
- **What can be said in the context of higher-order Sobolev spaces?**

Theorem (J. Kinnunen, 1997)

Let $1 < p \leq \infty$. If $f \in W^{1,p}(\mathbb{R}^d)$, then $\mathcal{M}f \in W^{1,p}(\mathbb{R}^d)$, and

$$|D_i \mathcal{M} f| \leq \mathcal{M}(D_i f), \quad i = 1, \ldots, d,$$

almost everywhere in $\mathbb{R}^d$. Thus, the operator $\mathcal{M} : W^{1,p}(\mathbb{R}^d) \to W^{1,p}(\mathbb{R}^d)$ is bounded.

Theorem (H. Tanaka, 2002)

If $f \in W^{1,1}(\mathbb{R})$, then the centered maximal function $\tilde{\mathcal{M}} f$ is weakly differentiable, and

$$\|(\tilde{\mathcal{M}} f)'\|_{L^1(\mathbb{R})} \leq 2\|f'\|_{L^1(\mathbb{R})}.$$

Theorem (Aldaz and Pérez Lázaro, 2007)

Let $f : \mathbb{R} \to \mathbb{R}$ be a function of bounded variation. Then $\tilde{\mathcal{M}} f$ is absolutely continuous, and

$$\mathrm{Var}(\tilde{\mathcal{M}} f) \leq \mathrm{Var}(f),$$

where $\mathrm{Var}(f)$ denotes the total variation of f.

Theorem (O. Kurka, 2015)

Let $f : \mathbb{R} \to \mathbb{R}$ be of bounded variation. Then

$$\mathrm{Var}(\mathcal{M} f) \leq 240004 \cdot \mathrm{Var}(f).$$

Theorem (Bober, Carneiro, Hughes, Pierce, 2012)

Let $f : \mathbb{Z} \to \mathbb{R}$ be a function of bounded variation. Then

$$\mathrm{Var}(\tilde{\mathbf{M}} f) \leq \mathrm{Var}(f).$$

Theorem (Temur, 2013)

Let $f : \mathbb{Z} \to \mathbb{R}$ be a function of bounded variation. Then

$$\mathrm{Var}(\mathbf{M} f) \leq C \cdot \mathrm{Var}(f),$$

where $C = (72000) \cdot 2^{12} + 4 = 294{,}912{,}004$.

- While significant progress has been made in understanding the behavior of **first derivatives** of maximal functions, much less is known about their **higher-order derivatives**.
- A first positive result in the discrete, non-centered setting arises when restricting to specific function classes.

Theorem (Temur, 2022)

Let $A \subseteq \mathbb{Z}$. For the characteristic function χ_A, we have

$$\|(\tilde{\mathbf{M}}\chi_A)''\|_{\ell^1} \leq 3\|\chi_A''\|_{\ell^1}.$$

Directions for Extension

The following are three natural directions for extending the current result:

(1) Can the estimate be extended to **other ℓ^p-norms, beyond** ℓ^1?

(2) Can similar bounds be established for **higher-order derivatives**?

(3) Can the result be generalized to **broader classes of functions** beyond characteristic functions?

Theorem (Temur, Ö., ∼2025)

Let $A \subseteq \mathbb{Z}$. For the characteristic function χ_A, we have

$$\|(\tilde{\mathbf{M}}\chi_A)''\|_{\ell^p} \leq 2^{1/p} \cdot 3\|\chi_A''\|_{\ell^p}.$$

This result extends the earlier ℓ^1-bound to a full range of ℓ^p-spaces.

Extension to Higher-Order Regularity via Induction

Assume that

$$\|\chi_A^{(k)}\|_{\ell^p} \leq C_k \|\chi_A^{(k+1)}\|_{\ell^p}$$

for all $k \geq 1$, where C_k is independent of $A \subseteq \mathbb{Z}$.

For any function $f : \mathbb{Z} \to \mathbb{R}$, we have

$$f^{(k+1)}(n) = f^{(k)}(n+1) - f^{(k)}(n) \quad \Rightarrow \quad \|f^{(k+1)}\|_{\ell^p} \leq 2\|f^{(k)}\|_{\ell^p}.$$

As an initial step in the induction argument: for $k = 2$, we already have

$$\|(\tilde{\mathbf{M}}\chi_A)''\|_{\ell^p} \leq 2^{1/p} \cdot 3\|\chi_A''\|_{\ell^p}.$$

Then, assuming the inductive hypothesis for k, we obtain the estimate for $k+1$:

$$\|(\tilde{\mathbf{M}}^d\chi_A)^{(k+1)}\|_{\ell^p} \leq 2\|(\tilde{\mathbf{M}}^d\chi_A)^{(k)}\|_{\ell^p} \leq 2C_{k,p}\|\chi_A^{(k)}\|_{\ell^p} \leq C_{k+1,p}\|\chi_A^{(k+1)}\|_{\ell^p},$$

where $C_{k+1,p} = 2C_{k,p}C_k$

Reducing the Problem to the Norm of χ_A

Main goal is to prove that

$$\|\chi_A^{(k)}\|_{\ell^p} \leq C_k \|\chi_A^{(k+1)}\|_{\ell^p}$$

for all $k \geq 1$, where C_k is independent of $A \subseteq \mathbb{Z}$.

Let $f : \mathbb{Z} \to \mathbb{C}$ be a finitely supported function. Its **Fourier transform**, denoted $\widehat{f} : \mathbb{T} \to \mathbb{C}$, is given by:

$$\widehat{f}(x) := \sum_{n \in \mathbb{Z}} f(n) e^{-2\pi i n x}.$$

Key Insight: The Fourier transform **diagonalizes** discrete differentiation. Specifically, for the k-th derivative:

$$\widehat{f^{(k)}}(x) = \left(e^{2\pi i x} - 1\right)^k \cdot \widehat{f}(x).$$

Application: Using the **Hausdorff-Young inequality**, we obtain for $1 \leq p \leq 2$:

$$\left\| \chi_A^{(k)} \right\|_{\ell^p(\mathbb{Z})} \geq \left\| \widehat{\chi_A^{(k)}} \right\|_{L^q(\mathbb{T})} = \left\| \left(e^{2\pi i x} - 1\right)^k \widehat{\chi_A} \right\|_{L^q(\mathbb{T})},$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Analysis Near the Maximum Point

Key Observation: The multiplier $\left|e^{2\pi ix} - 1\right| = 2|\sin(\pi x)|$ attains its maximum value of 2 at $x = 1/2$. We study its behavior near this critical point.

Local Expansion: For $x = 1/2 + t$ with $|t| < r$, we have

$$\left|e^{2\pi ix} - 1\right| = 2\left|\sin\left(\pi\left(\tfrac{1}{2} + t\right)\right)\right| = 2\cos(\pi t) \geq 2 - 2\pi^2 t^2.$$

Lower Bound: On the interval $E_r := (1/2 - r, 1/2 + r)$, we obtain

$$\left|\left(e^{2\pi ix} - 1\right)^k\right| \geq \left(2 - \pi^2 r^2\right)^k.$$

Final Estimate: Substituting into our earlier inequality yields:

$$\left\|\chi_A^{(k)}\right\|_{\ell^p(\mathbb{Z})} \geq \left(2 - \pi^2 r^2\right)^k \cdot \|\widehat{\chi_A}\|_{L^q(E_r)}.$$

Connecting Local and Global Norms via Nazarov–Turán

Key Objective: Relate the local norm $\|\widehat{\chi_A}\|_{L^q(E_r)}$ to the global norm $\|\widehat{\chi_A}\|_{L^q(\mathbb{T})}$

Nazarov–Turán Inequality. For any finite $A \subset \mathbb{Z}$ and measurable set $E \subset \mathbb{T}$ with $|E| > 0$:

$$\left\| \sum_{n\in A} a_n e^{2\pi inx} \right\|_{L^p(\mathbb{T})} \leq \left(\frac{14e}{|E|} \right)^{|A|-1} \left\| \sum_{n\in A} a_n e^{2\pi inx} \right\|_{L^p(E)}$$

Application:

- For $\widehat{\chi_A}(x) = \sum_{n\in A} e^{-2\pi inx}$
- With $E = E_r$ ($|E_r| = 2r$)

$$\|\widehat{\chi_A}\|_{L^q(\mathbb{T})} \leq \left(\frac{7e}{r} \right)^{|A|-1} \|\widehat{\chi_A}\|_{L^q(E_r)}$$

Final Estimate:

$$\left\| \chi_A^{(k)} \right\|_{\ell^p(\mathbb{Z})} \geq \left(2 - \pi^2 r^2 \right)^k \left(\frac{r}{7e} \right)^{|A|-1} \|\widehat{\chi_A}\|_{L^q(\mathbb{T})}$$

Optimal Parameter Choice: With $R = \frac{\sqrt{2}}{\pi}\left(\frac{|A|-1}{2k+|A|-1}\right)^{1/2}$, we obtain

$$\left\|\chi_A^{(k)}\right\|_{\ell^p(\mathbb{Z})} \geq \underbrace{\left(\frac{4k}{2k+|A|-1}\right)^k}_{\text{Size factor}} \underbrace{\left(\frac{\sqrt{2}}{7\pi e}\right)^{|A|-1}}_{\text{Constant}} \underbrace{\left(\frac{|A|-1}{2k+|A|-1}\right)^{\frac{|A|-1}{2}}}_{\text{Balance term}} \|\widehat{\chi_A}\|_{L^q(\mathbb{T})}.$$

When $2k \geq |A| - 1$,

$$\text{Balance term} \geq e^{-k} \quad \Rightarrow \quad \left\|\chi_A^{(k)}\right\|_{\ell^p(\mathbb{Z})} \geq 2^k(7\pi)^{-2k}e^{-3k}\|\widehat{\chi_A}\|_{L^q(\mathbb{T})}.$$

Special Cases:

- **Plancherel case** ($p = 2$): $\|\widehat{\chi_A}\|_{L^2(\mathbb{T})} = \sqrt{|A|}$
- **Extreme case** ($p = 1$): $\|\widehat{\chi_A}\|_{L^\infty(\mathbb{T})} \geq \sqrt{|A|}$
  Leading to

$$\left\|\chi_A^{(k)}\right\|_{\ell^1(\mathbb{Z})} \geq 2^k(7\pi)^{-2k}e^{-3k}\sqrt{|A|}$$

Hölder Interpolation: For any finite set $A \subset \mathbb{Z}$ and $1 \leq p \leq \infty$:

$$\left\|\chi_A^{(k)}\right\|_{\ell^1} \leq \left\|\chi_A^{(k)}\right\|_{\ell^p} \cdot |A|^{1-1/p} \cdot (k+1)^{1-1/p}$$

When $2k \geq |A| - 1$:

$$\left\|\chi_A^{(k)}\right\|_{\ell^p} \geq \frac{2^k}{(7\pi)^{2k} e^{3k}} \cdot \frac{|A|^{1/p-1/2}}{(k+1)^{1-1/p}}$$

Maximal Operator Estimate:

$$\begin{aligned}
\left\|\left(\widetilde{\mathbf{M}}\chi_A\right)^{(k)}\right\|_{\ell^p(\mathbb{Z})} &\leq 2^k \left\|\widetilde{\mathbf{M}}\chi_A\right\|_{\ell^p(\mathbb{Z})} \\
&\leq 2^k \mathcal{C}_p \|\chi_A\|_{\ell^p(\mathbb{Z})} \\
&= 2^k \mathcal{C}_p \, |A|^{1/p} \\
&\leq \underbrace{2^{2k}(7\pi)^{2k} e^{3k} (k+1)^{1-1/p} \mathcal{C}_p}_{\mathcal{C}_{p,k}} \left\|\chi_A^{(k)}\right\|_{\ell^p}
\end{aligned}$$

Theorem (Temur & Özcan, 2025)

For any finite subset $A \subset \mathbb{Z}$ satisfying $2k \geq |A| - 1$, the discrete Hardy-Littlewood maximal function satisfies:

$$\|(\widetilde{\mathbf{M}}\chi_A)^{(k)}\|_{\ell^p(\mathbb{Z})} \leq C_{k,p}\|\chi_A^{(k)}\|_{\ell^p(\mathbb{Z})}$$

with constant

$$C_{k,p} = 2^{2k}(7\pi)^{2k}e^{3k}(k+1)^{1-1/p}\mathcal{C}_p.$$

We also recetly have revomed restrictions on derivative order k!

Temur, F., Özcan, H.B. (2025). Higher Regularity of Discrete Maximal Functions. arXiv:2504.13019

An Application in Image Processing: Max Pooling VS. Local Averaging

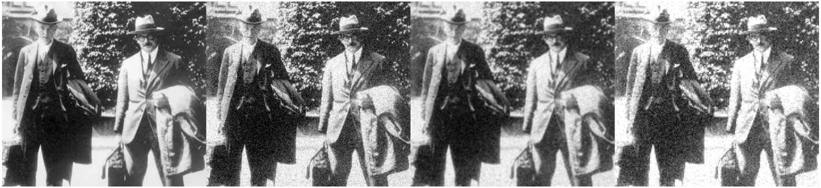


Figure: Max pooling vs. average pooling

Aspect	Max Pooling	Average Pooling
Definition	Takes the maximum mean in each region	Takes the mean in each region
Edge Preservation	Preserves sharp edges	Smooths edges and textures
Noise Suppression	Less smoothing of noise	Stronger smoothing, may blur details
Use Case	Object recognition, edge detection	Denoising, generalization

Further details will be discussed in upcoming seminars.

Thank You!

Questions & Discussion

hikmetozcan@iyte.edu.tr

Department of Mathematics, IYTE