Krull-Schmidt-Remak-Azumaya Theorem

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Overview

Introduction

Motivation Examples

- The fundamental theorem of arithmatic states that every positive integer can be written as a unique product of prime numbers up to ordering.
- The fundamental theorem of finite abelian groups states that every finite abelian group G can be expressed (uniquely) as the direct sum of cyclic subgroups of prime-power order.

Question

- It is natural to ask that a decomposition of an arbitray module into indecomposable summands (if exists) is unique up to isomorphism.
- KSRA gives an affirmative answer for this natural question.

Theorem (Krull–Schmidt–Remak–Azumaya)

Let M be a module that is a direct sum of modules with local endomorphism rings. Then, any two direct sum decompositons of M into indecomposable direct summands are isomorphic.

Characterization of Local Rings

Definition

Let R be an arbitrary ring with unity and $A = \{a \in A \mid a \text{ is non invertible}\}$. R is said to be a **local** ring if one of the following equivalent properties is satisfied:

1. A is additively closed.
2. A is a two sided ideal.
3. A is the largest proper right ideal.
4. A is the largest proper left ideal.
5. In R there exists a largest proper right ideal.
6. In R there exists a largest proper left ideal.
7. For every $r \in R$ either r or $1 - r$ is right invertible.
8. For every $r \in R$ either r or $1 - r$ is left invertible.
9. For every $r \in R$ either r or $1 - r$ is invertible.

Example

The power series ring $K[[x]] = \{\sum_{n=0}^{\infty} a_n x^n \mid a_n \in K\}$ over a field K is local. The non-invertible elements are precisely those with constant term zero and the set of these elements is additively closed.

Example

Localizations of commutative rings at prime ideals are local.
$R_{(P)} = \{\frac{r}{a} \mid r \in R \wedge a \in R \setminus P\}$ is a ring. The elements of the form $\frac{r}{a}$ with $r \in P$ are non-invertible. The set of these elements is additively closed and consequently $R_{(P)}$ is a local ring.

Definition

In a ring R an element $e \in R$ is said to be *idempotent* if $e^2 = e$.

Proposition

If R is a local ring, then R has only trivial idempotents 0 and 1.

Proposition

The following are equivalent for a ring R:

1. *R_R is indecomposable.*
2. *${}_R R$ is indecomposable.*
3. *R has the only trivial idempotents.*

Theorem

Let $S = End(M_R)$, then the following are equivalent:

1. *M_R is indecomposable.*
2. *S_S is indecomposable.*
3. *${}_S S$ is indecomposable.*
4. *S has the only trivial idempotents.*

Corollary

If $S = End(M_R)$ is a local ring, then M is indecomposable.

Exchange Property

Definition

Given a cardinal $\aleph$, an R-module M is said to have the $\aleph$-exchange property if for any R-module G and any two decompositions

$$G = M' \oplus N = \bigoplus_{i \in I} A_i,$$

where $M' \cong M$ and $|I| \leq \aleph$, there are R-submodules B_i of A_i for every $i \in I$ such that

$$G = M' \oplus \left(\bigoplus_{i \in I} B_i \right).$$

We say that M has exchange property if M has $\aleph$-exchange property for every cardinal $\aleph$.

Proposition

Let $M = M_1 \oplus M_2$ be a module. M has the exhange property if and only if M_1 and M_2 have the exchange property.

Proposition

If a module M has 2-exchange property, then it has finite exchange property.

Theorem

Let M be an indecomposable right R-module. Then the following properties are equivalent:

1. *$End(M_R)$ is local.*
2. *M has the finite exchange property.*
3. *M has the exchange property.*

Lemma

Let M be a module with the $\aleph$-exchange property and let $M = \bigoplus_{i \in I} A_i = \bigoplus_{j \in J} B_j$ with I is finite and $|J| \leq \aleph$. If each A_i and B_i are indecomposable, then these two direct sum decompositions of M are isomorphic.

Theorem (Krull–Schmidt–Remak–Azumaya)

Let M be a module that is a direct sum of modules with local endomorphism rings. Then, any two direct sum decompositons of M into indecomposable direct summands are isomorphic.

Idea of the proof:

Let $M = \bigoplus_{i \in I} M_i = \bigoplus_{j \in J} N_j$, where $End(M_i)$ is local and N_j is indecomposable.
Consider an indecomposable module A and define

$$I_A = \{i \in I | M_i \cong A\}$$

and

$$J_A = \{j \in J | N_j \cong A\}.$$

Aim: $|I_A| = |J_A|$.

Case 1: $|I_A| < \infty$

Let $M(I) := \bigoplus_{i \in I} M_i$ and $N(J) := \bigoplus_{j \in J} N_j$. Then,

$$M(I_A) \oplus M(I - I_A) = N(J)$$

and $M(I_A)$ has the exhange property. So,

$$M(I_A) \oplus N(J') = N(J),$$

where $N(J) = N(J') \oplus N(J - J')$ and $M(I_A) \cong N(J - J')$.
(Aim: $J - J' = J_A$)

$$M(I_A) \cong N(J - J')$$

Hence,

$$\theta : I_A \longrightarrow (J - J')$$

is a bijection s.t. $M_i \cong N_{\theta(i)}$ which means that $(J - J') \subseteq J_A$.

Conversely, $M(I - I_A)$ does not have a direct summand which is isomorphic to A.
Since $M(I - I_A) \cong N(J')$, $J_A \subseteq (J - J')$. As a resut $J_A = (J - J')$ which implies that $M(I) \cong N(J)$ when $|I_A|$ is finite.

Q.E.D.

Corollary (1)

Any two direct sum decompositions of a semisimple module into simple direct summands are isomorphic.

Corollary (2)

If M_R is an R-module that is a direct sum of artinian modules with simple socle. Then any two direct sum decompositions of M_R into indecomposable summands are isomorphic.

Corollary (3)

Let M_R be an R-module that is a direct sum of noetherian local modules. Then any two direct sum decompositions of M_R into indecomposable summands are isomorphic.

Open Question 1

Does a module with the finite exchange property have the exchange property?

Open Question 2

Suppose that M_R is a right R-module satisfying the assumption of KSRA Theorem and $M = N \oplus N'$.
Does N also satisfy the assumption of KSRA Theorem?
In other words, is N a direct sum of modules with local endomorphism rings?

References

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Thank $\bigoplus$ *You*