Higher Regularity of Discrete Hardy–Littlewood Maximal Function

İzmir Mathematics Days 7

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The Hardy–Littlewood Maximal Function

For a locally integrable function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ or $\mathbb{C}$, the **centered Hardy–Littlewood maximal function** $\mathcal{M}f$ is defined as

$$\mathcal{M}f(x) \;=\; \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy,$$

where $B(x,r)$ is the ball centered at x with radius r.

The **non-centered Hardy–Littlewood maximal function** $\widetilde{\mathcal{M}}f$ is defined as

$$\widetilde{\mathcal{M}}f(x) \;=\; \sup_{B \ni x} \frac{1}{|B|} \int_{B} |f(y)|\, dy,$$

where the supremum is taken over all balls B containing x.

Discrete Hardy–Littlewood Maximal Function

Let $f : \mathbb{Z} \longrightarrow \mathbb{R}$ or $\mathbb{C}$. The **centered discrete Hardy-Littlewood maximal function Mf** is defined as

$$\mathbf{M}f(n) \;=\; \sup_{r\in\mathbb{Z}^+} \frac{1}{2r+1} \sum_{k=-r}^{r} |f(n+k)|,$$

where the average is taken over the discrete interval $[n-r, n+r]$.

The **non-centered discrete Hardy-Littlewood maximal function** is defined as

$$\widetilde{\mathbf{M}}f(n) \;=\; \sup_{\substack{a\leq n\leq b\\ a,b\in\mathbb{Z}}} \frac{1}{b-a+1} \sum_{k=a}^{b} |f(k)|,$$

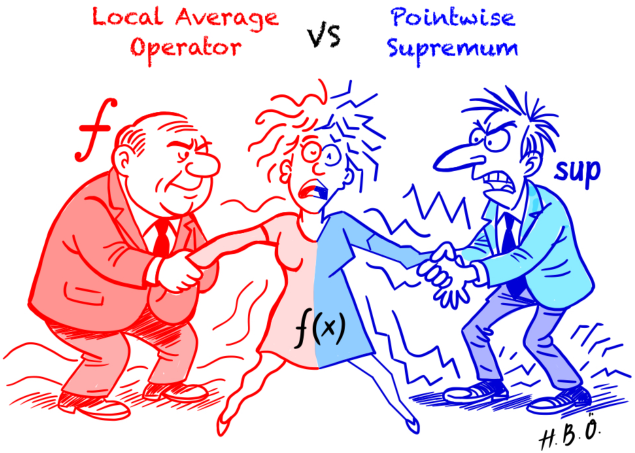
where the supremum is taken over all discrete intervals containing n.

Philosophy of the Regularity Theory of Maximal Operators

$$\mathcal{M}f(x) = \underbrace{\sup_{r>0}}_{\text{Pointwise Supremum}} \underbrace{\frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|\,dy}_{\text{Local Average}}$$

Local Averaging *(Smoothing)*	**Pointwise Supremum** *(Irregularity)*
▶ Tends to **improve regularity** ▶ Reduces sharp oscillations ▶ Preserves general trend of the function	▶ Can **increase irregularity** ▶ Highlights spikes and extremes ▶ May amplify oscillations

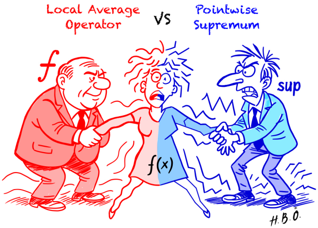
Who wins?



The Challenge in Regularity Theory

The **regularity theory** of **maximal operators** investigates:

- How the **maximal operator** interacts with **function spaces**.
- Whether it **maintains**, **refines**, or **deteriorates** the **regularity** properties (continuity, differentiability, smoothness) of functions.
- If the **discrete derivative** of the maximal operator can be bounded in terms of the original function.



Who wins?

Literature Review: Continuous Setting

Kinnunen (1997): For $1 < p \leq \infty$, if $f \in W^{1,p}(\mathbb{R}^d)$, then $\mathcal{M}f \in W^{1,p}(\mathbb{R}^d)$, and

$$|D_i \mathcal{M} f| \leq \mathcal{M}(D_i f), \quad i = 1, \dots, d.$$

Tanaka (2002): If $f \in W^{1,1}(\mathbb{R})$, the non-centered maximal function $\widetilde{\mathcal{M}}f$ satisfies

$$\|(\widetilde{\mathcal{M}}f)'\|_{L^1(\mathbb{R})} \leq 2\,\|f'\|_{L^1(\mathbb{R})}.$$

Aldaz & Pérez Lázaro (2007): For $f : \mathbb{R} \to \mathbb{R}$ of bounded variation, $\widetilde{\mathcal{M}}f$ is an absolutely continuous function and

$$\mathrm{Var}(\widetilde{\mathcal{M}}f) \leq \mathrm{Var}(f).$$

Kurka (2015): Let $f : \mathbb{R} \to \mathbb{R}$ be a function of bounded variation. Then

$$\mathrm{Var}(\mathcal{M}f) \leq 240{,}004 \cdot \mathrm{Var}(f).$$

Literature Review: Discrete Setting (First-Order)

Bober, Carneiro, Hughes, Pierce (2012): Let $f : \mathbb{Z} \to \mathbb{R}$ be a function of bounded variation. Then

$$\mathrm{Var}(\widetilde{\mathbf{M}} f) \leq \mathrm{Var}(f).$$

Temur (2013): Let $f : \mathbb{Z} \to \mathbb{R}$ be a function of bounded variation. Then

$$\mathrm{Var}(\mathbf{M} f) \leq C \cdot \mathrm{Var}(f),$$

where $C = (72000) \cdot 2^{12} + 4 = 294{,}912{,}004$.

Literature Review: Discrete Higher-Order Case

Temur (2022) proved that for a finite subset $A \subseteq \mathbb{Z}$ and its characteristic function χ_A,

$$\|(\tilde{\mathbf{M}}\chi_A)''\|_{\ell^p(\mathbb{Z})} \;\le\; C_p\,\|\chi_A''\|_{\ell^p(\mathbb{Z})},$$

where

$$C_p = 3^{1/p}\,2^{\,1-1/p}, \quad (1 \le p < \infty), \qquad C_\infty = 2.$$

- First positive result on higher-order derivatives in the discrete, non-centered setting.
- Shows that discrete non-centered maximal operators preserve second-order regularity for characteristic functions.

Question in Higher-Order Regularity

Can we establish similar bounds for higher-order derivatives?

Attempt for Third Derivative

We consider the third discrete derivative of the maximal function applied to a characteristic function:

$$\left\| \left(\widetilde{\mathbf{M}} \chi_A \right)^{(3)} \right\|_{\ell^1} = \sum_{n \in \mathbb{Z}} \left| \widetilde{\mathbf{M}} \chi_A(n+3) - 3\widetilde{\mathbf{M}} \chi_A(n+2) + 3\widetilde{\mathbf{M}} \chi_A(n+1) - \widetilde{\mathbf{M}} \chi_A(n) \right|$$

$$= \sum_{n \in \mathbb{Z}} \left| \underbrace{\left[\widetilde{\mathbf{M}} \chi_A(n+3) - 2\widetilde{\mathbf{M}} \chi_A(n+2) + \widetilde{\mathbf{M}} \chi_A(n+1) \right]}_{\text{shifted 2nd derivative}} - \underbrace{\left[\widetilde{\mathbf{M}} \chi_A(n+2) - 2\widetilde{\mathbf{M}} \chi_A(n+1) + \widetilde{\mathbf{M}} \chi_A(n) \right]}_{\text{previous 2nd derivative}} \right|$$

$$= \sum_{n \in \mathbb{Z}} \left| \left(\widetilde{\mathbf{M}} \chi_A \right)''(n+1) - \left(\widetilde{\mathbf{M}} \chi_A \right)''(n) \right|$$

$$\leq 2 \left\| \left(\widetilde{\mathbf{M}} \chi_A \right)'' \right\|_{\ell^1} .$$

Key Idea: The third derivative can be expressed as the difference of two shifted second derivatives:

$$a - 3b + 3c - d = (a - 2b + c) - (b - 2c + d)$$

General Observation: Higher-Order Derivatives

Can we estimate the higher order derivatives via lower order derivatives?

In the discrete setting, the k-th order derivative satisfies the recurrence:

$$f^{(k)}(n) = f^{(k-1)}(n+1) - f^{(k-1)}(n), \quad k \geq 1, \quad f^{(0)} = f.$$

Applying the triangle inequality, we obtain a pointwise bound:

$$|f^{(k)}(n)| \leq |f^{(k-1)}(n+1)| + |f^{(k-1)}(n)|.$$

Consequently, for any $1 \leq p < \infty$, Minkowski's inequality implies:

$$\|f^{(k)}\|_{\ell^p(\mathbb{Z})} \leq 2\,\|f^{(k-1)}\|_{\ell^p(\mathbb{Z})}.$$

Higher-Order Regularity Estimate

Applying the recursive bound together with Temur's second-order result, we have

$$\left\| \left(\widetilde{\mathbf{M}} \chi_A \right)^{(k)} \right\|_{\ell^p(\mathbb{Z})} \leq 2^{k-2} \left\| \left(\widetilde{\mathbf{M}} \chi_A \right)'' \right\|_{\ell^p(\mathbb{Z})} \leq C_{k,p} \left\| \chi_A'' \right\|_{\ell^p(\mathbb{Z})}, \quad k \geq 2.$$

To express this in terms of the k-th derivative, we require

$$\left\| \chi_A'' \right\|_{\ell^p(\mathbb{Z})} \lesssim_{k,p} \left\| \chi_A^{(k)} \right\|_{\ell^p(\mathbb{Z})}.$$

If the above holds, we obtain the **desired estimate**

$$\left\| \left(\widetilde{\mathbf{M}} \chi_A \right)^{(k)} \right\|_{\ell^p(\mathbb{Z})} \lesssim_{k,p} \left\| \chi_A^{(k)} \right\|_{\ell^p(\mathbb{Z})}.$$

Since first-order regularity is already established, our **main goal** reduces to

$$\|\chi_A'\|_{\ell^p(\mathbb{Z})} \lesssim_{k,p} \|\chi_A^{(k)}\|_{\ell^p(\mathbb{Z})}.$$

Fourier Analytic Approach

Let $f : \mathbb{Z} \to \mathbb{C}$ be finitely supported. Its **Fourier transform**, $\widehat{f} : \mathbb{T} \to \mathbb{C}$, is defined by

$$\widehat{f}(x) := \sum_{n \in \mathbb{Z}} f(n)\, e^{-2\pi i n x}.$$

Key fact: The Fourier transform **diagonalizes** discrete differentiation:

$$\widehat{f^{(k)}}(x) = \left(e^{2\pi i x} - 1\right)^k \widehat{f}(x).$$

By the **Hausdorff–Young inequality**, in the case $p = 1$ (so $q = \infty$),

$$\|\chi_A^{(k)}\|_{\ell^1(\mathbb{Z})} \;\geq\; \|\widehat{\chi_A^{(k)}}\|_{L^\infty(\mathbb{T})} = \sup_{x \in \mathbb{T}} \left| \left(e^{2\pi i x} - 1\right)^k \widehat{\chi_A}(x) \right|.$$

Consider the multiplier

$$m(x) = e^{2\pi i x} - 1.$$

- Its modulus is

$$|m(x)| = |e^{2\pi i x} - 1| = 2|\sin(\pi x)|.$$

- The maximum value is attained at

$$x = \tfrac{1}{2} + n, \quad n \in \mathbb{Z}, \qquad |m(\tfrac{1}{2})| = 2.$$

- Near $x = \frac{1}{2}$, writing $x = \frac{1}{2} + t$ with small t, we have

$$|m(x)| = 2|\cos(\pi t)| \;\geq\; 2 - \pi^2 t^2.$$

Thus, on $E_r = \left(\frac{1}{2} - r, \frac{1}{2} + r\right)$, for sufficiently small $r > 0$,

$$\|\chi_A^{(k)}\|_{\ell^1(\mathbb{Z})} \;\geq\; \sup_{x \in E_r} \left|(e^{2\pi i x} - 1)^k \, \widehat{\chi_A}(x)\right| \;\geq\; (2 - \pi^2 r^2)^k \sup_{x \in E_r} |\widehat{\chi_A}(x)|.$$

Connecting Local and Global Norms via Nazarov–Turán

To estimate $\|\widehat{\chi_A}\|_{L^\infty(E_r)}$, we apply the **Nazarov–Turán inequality**.

Nazarov–Turán Inequality. Let

$$P(x) = \sum_{n \in A} a_n e^{2\pi i n x}$$

be a trigonometric polynomial associated with a finite, nonempty set $A \subset \mathbb{Z}$, with coefficients $a_n \in \mathbb{C}$. For any measurable $E \subset \mathbb{T}$,

$$\|P\|_{L^\infty(\mathbb{T})} \le \left(\frac{14e}{|E|}\right)^{|A|-1} \|P\|_{L^\infty(E)}.$$

Applying this to $P(x) = \widehat{\chi_A}(-x)$ with $|E_r| = 2r$,

$$\|\widehat{\chi_A}\|_{L^\infty(E_r)} \ge \left(\frac{r}{7e}\right)^{|A|-1} \|\widehat{\chi_A}\|_{L^\infty(\mathbb{T})} \ge \left(\frac{r}{7e}\right)^{|A|-1} |\widehat{\chi_A}(0)| = \left(\frac{r}{7e}\right)^{|A|-1} |A|.$$

Thus, we obtain

$$\|\chi_A^{(k)}\|_{\ell^1(\mathbb{Z})} \ge (2 - \pi^2 r^2)^k \left(\frac{r}{7e}\right)^{|A|-1} |A|.$$

Optimal Parameter Choice. Choosing

$$r_* = \frac{\sqrt{2}}{\pi}\sqrt{\frac{|A|-1}{2k+|A|-1}},$$

we obtain the lower bound

$$\|\chi_A^{(k)}\|_{\ell^1(\mathbb{Z})} \;\geq\; 2^{\,k+(|A|-1)/2}(7e\pi)^{-|A|+1}\left(\frac{2k}{2k+|A|-1}\right)^k\left(\frac{|A|-1}{2k+|A|-1}\right)^{(|A|-1)/2}|A|.$$

Simplification for $2k \geq |A|-1$:

$$\|\chi_A^{(k)}\|_{\ell^1(\mathbb{Z})} \;\geq\; 2^{(|A|-1)/2}(7\pi)^{-2k}e^{-3k}\,|A|.$$

From ℓ^1 to ℓ^p Estimates

The support of $\chi_A^{(k)}$ satisfies

$$\operatorname{supp}(\chi_A^{(k)}) \subset \bigcup_{j=0}^{k} (A - j).$$

This implies

$$|\operatorname{supp}(\chi_A^{(k)})| \leq (k+1)|A|.$$

By Hölder's inequality,

$$\|\chi_A^{(k)}\|_{\ell^1} \leq \|\chi_A^{(k)}\|_{\ell^p} \cdot [(k+1)|A|]^{1/q}.$$

Combining with the previous ℓ^1 bound, we obtain

$$\|\chi_A^{(k)}\|_{\ell^p} \gtrsim 2^{(|A|-1)/2} (7\pi)^{-2k} e^{-3k} (k+1)^{-1/q} \|\chi_A\|_{\ell^p}.$$

Main Result

Theorem (Temur & Ö., 2025)

Let $A \subset \mathbb{Z}$ be a finite set with $2k \geq |A| - 1$. Then the k-th discrete derivative satisfies

$$\|\chi_A^{(k)}\|_{\ell^p(\mathbb{Z})} \;\geq\; (7\pi)^{-2k} e^{-3k} (k+1)^{-1/q} \|\chi_A\|_{\ell^p(\mathbb{Z})},$$

where q is the Hölder conjugate of p, i.e., $\frac{1}{p} + \frac{1}{q} = 1$ (with $\frac{1}{\infty} = 0$).

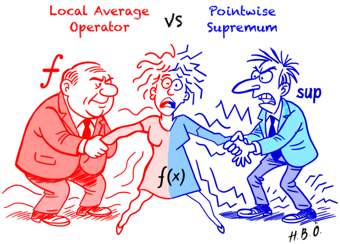
Corollary

Let $A \subset \mathbb{Z}$ be finite, and let χ_A be its characteristic function. Then for any $k \geq 1$ and $1 \leq p \leq \infty$,

$$\|(\mathbf{M}\chi_A)^{(k)}\|_{\ell^p(\mathbb{Z})} \lesssim_{k,p} \|\chi_A^{(k)}\|_{\ell^p(\mathbb{Z})}.$$

Temur, F., Özcan, H.B. (2025). Higher Regularity of Discrete Maximal Functions.
arXiv:2504.13019

Thank You!



Who wins?

The good guys always win.