Discretization Approach to the Weak Boundedness of Maximal Convolution Operators

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A Short Preliminary

The **convolution** $f * g$ of two functions f and g is defined as the integral of the product of the two functions after one is reflected about the y-axis and shifted:

$$(f * g)(x) := \int_{-\infty}^{\infty} f(x-y)g(y)\,dy.$$

Alternatively, the convolution can be expressed as

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\,dy.$$

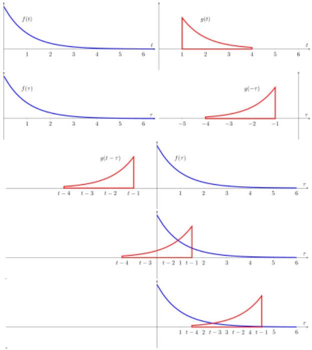

Figure: Illustration of the convolution operation.

A Short Preliminary

The **Dirac delta function** (or δ-distribution) can be heuristically represented as

$$\delta(x) = \begin{cases} 0, & \text{if } x \neq 0 \\ \infty, & \text{if } x = 0 \end{cases}$$

and satisfies

$$\int_{-\infty}^{\infty} \delta(x)\, dx = 1.$$

- **Shifting Property:**

$$(f * \delta)(x) = f(x),$$

which indicates that convolution with the delta function shifts the function.

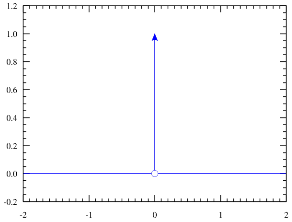

Figure: The Dirac delta $\delta(x)$, illustrating its behavior as an impulse.

A Short Preliminary

- **Strong Type (p, p) Inequality**

  An operator T is said to satisfy a *strong type* (p, p) inequality for $1 \leq p \leq \infty$ if there exists a constant $C > 0$ such that for all functions f in the domain of T

  $$\|Tf\|_p \leq C\|f\|_p,$$

  where $\|\cdot\|_p$ denotes the L^p-norm. This means that T is a bounded operator on L^p functions.

- **Weak Type (p, p) Inequality**

  An operator T is said to satistfy a *weak type* (p, p) inequality for $1 \leq p < \infty$ if there exists a constant $C > 0$ such that for all functions f and for all $\lambda > 0$,

  $$|\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}| \leq \frac{C\|f\|_p^p}{\lambda^p}.$$

  This condition controls the measure of the level sets where the output of the operator exceeds λ. Thus, T is weakly bounded on L^p functions.

Hardy-Littlewood Maximal Function

The **Hardy-Littlewood maximal function** $\mathcal{M}$ of a locally integrable function f on $\mathbb{R}^d$ is defined as

$$\mathcal{M}f(x) := \sup_{r>0} \underbrace{\frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy}_{\mathcal{A}_r|f|(x)},$$

where $\mathcal{A}_r|f|(x)$ denotes the average of $|f|$ over the ball $B(x,r)$ centered at x with radius r, and $|B(x,r)|$ is the Lebesgue measure (volume) of the ball.

For the **one-dimensional case**, the Hardy-Littlewood maximal function takes the form

$$\mathcal{M}f(x) := \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} |f(y)|\, dy,$$

where the supremum is taken over all intervals centered at x with radius r.

Types of Hardy-Littlewood Maximal Function

In the one-dimensional case, we distinguish the following types of Hardy-Littlewood maximal functions:

- **Non-centered Hardy-Littlewood maximal function** $\tilde{\mathcal{M}}$:

$$\tilde{\mathcal{M}}f(x) = \sup_{r,s\geq 0} \frac{1}{r+s} \int_{x-r}^{x+s} |f(y)|\, dy.$$

  This version takes the supremum over all intervals with center x and varying radii r and s.

- **Truncated Hardy-Littlewood maximal function** $\mathcal{M}^\tau$:

$$\mathcal{M}^\tau f(x) = \sup_{0<r\leq\tau} \frac{1}{2r} \int_{x-r}^{x+r} |f(y)|\, dy.$$

  Here, the supremum is taken over intervals with radii up to a truncation parameter τ.

Types of Hardy-Littlewood Maximal Function

- **Right-sided Hardy-Littlewood maximal function** $\mathcal{M}^{\rightarrow}$:

$$\mathcal{M}^{\rightarrow} f(x) = \sup_{h \geq 0} \frac{1}{h} \int_{x}^{x+h} |f(y)| \, dy.$$

- **Left-sided Hardy-Littlewood maximal function** $\mathcal{M}^{\leftarrow}$:

$$\mathcal{M}^{\leftarrow} f(x) = \sup_{h \geq 0} \frac{1}{h} \int_{x-h}^{x} |f(y)| \, dy.$$

Weak-Type $(1,1)$ Inequality for $\mathbf{M}^{\rightarrow} f$

Theorem: The one-sided maximal operator

$$\mathbf{M}^{\rightarrow} f(x) = \sup_{h>0} \frac{1}{h} \int_{x}^{x+h} |f(y)| \, dy$$

satisfies the weak-type $(1,1)$ inequality, i.e.,

$$|\{x \in \mathbb{R}^n : \mathbf{M}^{\rightarrow} f(x) > \lambda\}| \leq \frac{C}{\lambda} \|f\|_1$$

for some constant $C > 0$ independent of f and λ.

Proof (via Covering Argument):

Define the set $E_\lambda = \{x \in \mathbb{R}^n : \mathbf{M}^{\rightarrow} f(x) > \lambda\}$.

We aim to show that

$$|E_\lambda| \leq \frac{C}{\lambda} \|f\|_{\mathcal{L}^1},$$

where C is a constant.

Step 1: Selection of Intervals

For $x \in E_\lambda$, we have $\mathbf{M}^{\rightarrow} f(x) > \lambda$. Therefore, there exists a value $h_x > 0$ such that

$$\frac{1}{h_x} \int_x^{x+h_x} |f(y)|\, dy > \lambda.$$

This implies

$$\int_x^{x+h_x} |f(y)|\, dy > \lambda h_x.$$

Consider the collection of intervals $\left\{[x, x+h_x] : x \in E_\lambda\right\}$, which covers the level set E_λ.

Step 2: Vitali Covering Lemma

The intervals in the collection $\left\{[x, x+h_x] : x \in E_\lambda\right\}$ may overlap, so we apply the **Vitali covering lemma**, which guarantees the existence of a disjoint subcollection $\{I_i\} \subset \left\{[x, x+h_x] : x \in E_\lambda\right\}$ such that

$$E_\lambda \subseteq \bigcup_i I_i.$$

Step 3: Estimate the Measure

Note that each interval I_i satisfies

$$\int_{I_i} |f(y)|\, dy \geq \lambda |I_i|.$$

Since the intervals $\{I_i\}$ are disjoint, we can sum their lengths without overlap:

$$\lambda \sum_i |I_i| \leq \sum_i \int_{I_i} |f(y)|\, dy \leq \int_{\mathbb{R}} |f(y)|\, dy = \|f\|_1.$$

Since $E_\lambda \subseteq \bigcup_i I_i$, we have

$$|E_\lambda| \leq \sum_i |I_i|,$$

which gives

$$|E_\lambda| \leq \sum_{i \in I} |I_i| \leq \frac{\|f\|_1}{\lambda},$$

where $C = 1$. □

Maximal Functions of Convolution Type

The types of Hardy-Littlewood maximal functions $\mathcal{M}$ can be interpreted as the convolution of $|f|$ with a suitable characteristic function:

- **Centered Hardy-Littlewood maximal function** $\tilde{\mathcal{M}}$:

$$\mathcal{M}f(x) = \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} |f(y)|\, dy = \sup_{r\geq 0} \left(\chi_{[-r,r]}(x) * |f|\right),$$

where $\chi_{[-r,r]}$ is the characteristic function of the interval $[-r, r]$.

- **One-sided Hardy-Littlewood maximal function** $\tilde{\mathcal{M}}$:

$$\mathbf{M}^{\rightarrow} f(x) = \sup_{h>0} \frac{1}{h} \int_{x}^{x+h} |f(y)|\, dy = \sup_{h\geq 0} \left(\chi_{[-h,0]}(x) * |f|\right),$$

where $\chi_{[-h,0]}$ is the characteristic function of the interval $[-h, 0]$.

Maximal Convolution Operator

Let $\{k_j\} \subset \mathcal{L}^1(\mathbb{R}^n)$ be a sequence of positive functions, and let $f \in \mathcal{L}^1(\mathbb{R}^n)$. The **maximal convolution operator** is defined as

$$\mathbf{K}^* f(x) := \sup_j |k_j * f(x)|.$$

In particular, by taking the kernels $k_h = \frac{1}{h}\chi_{[-h,0]}$, the one-sided Hardy-Littlewood maximal function $\mathbf{M}^{\rightarrow}$ can be expressed as a maximal convolution operator:

$$\mathbf{M}^{\rightarrow} f(x) = \sup_{h>0} (k_h * |f|)(x).$$

Discretization Method: Key Equivalence Result

Theorem: The following conditions are equivalent:

(a) The operator $\mathbf{K}^*$ satisfies the weak-type $(1, 1)$ inequality, that is,

$$\left|\left\{x \in \mathbb{R}^n : \mathbf{K}^* f(x) > \lambda\right\}\right| \leq \frac{\mathbf{C}}{\lambda}\|f\|_1,$$

for some constant $\mathbf{C} > 0$ independent of $f \in \mathcal{L}^1(\mathbb{R}^n)$ and $\lambda > 0$.

(b) For any finite set of points $\{a_j\}_{j=1}^N \subset \mathbb{R}^n$ (not necessarily distinct) and any $\lambda > 0$, there exists a constant $\mathbf{C} > 0$ such that

$$\left|\left\{x \in \mathbb{R}^n : \sup_j \left|\sum_{k=1}^N k_j(x - a_k)\right| > \lambda\right\}\right| \leq \frac{\mathbf{C}}{\lambda} N.$$

Moreover, the same constant $\mathbf{C}$ can be used in both (a) and (b).

M. Trinidad Menarguez, F. Soria, Weak Type $(1, 1)$ *Inequalities of Maximal Convolution Operators, Rendiconti Del Circolo Matematico Di Palermo, Serie II, Tomo XLI (1992) pp. 342-352.*

Proof (via Discretization Method):

We aim to show that for any $\lambda > 0$,

$$|\{x \in \mathbb{R} : \mathbf{M}^{\rightarrow} f(x) > \lambda\}| \leq \frac{1}{\lambda} \|f\|_1.$$

Using the key equivalence result (**discretization method**), it is equivalent to prove that for any finite set of points $\{a_k\}_{k=1}^{N} \subset \mathbb{R}$ (not necessarily distinct) and $\lambda > 0$, we have

$$\left| \left\{ x \in \mathbb{R} : \sup_{h>0} \left| \sum_{k=1}^{N} k_h(x - a_k) \right| > \lambda \right\} \right| \leq \frac{N}{\lambda},$$

where the kernels $k_h = \frac{\chi_{[-h,0]}}{h}, \quad h > 0.$

Let $\mu := \sum_{k=1}^{N} \delta_{a_k}$, where $\delta_{a_k}(x) = \delta(x - a_k)$ is the Dirac delta centered at a_k. Notice that

$$\begin{aligned} \mathbf{M}^{\rightarrow}\mu(x) &= \sup_{h>0}(k_h * \mu)(x) \\ &= \sup_{h>0}\left(k_h * \sum_{k=1}^{N} \delta_{a_k}\right)(x) \\ &= \sup_{h>0}\sum_{k=1}^{N}(k_h * \delta_{a_k})(x) \\ &= \sup_{h>0}\sum_{k=1}^{N} k_h(x - a_k). \end{aligned}$$

Thus, we need to show that

$$\left|\left\{x \in \mathbb{R} : \sup_{h>0}\left|\sum_{k=1}^{N} k_h(x - a_k)\right| > \lambda\right\}\right| = \left|\{x \in \mathbb{R} : \mathbf{M}^{\rightarrow}\mu(x) > \lambda\}\right| \leq^{1} \frac{N}{\lambda}.$$

We proceed by induction on N to prove the inequality [1].

[1] **We aim to show this inequality**

- **Initial Step** $N = 1$: In this case, we obtain

$$\begin{aligned}
\mathbf{M}^{\rightarrow}\mu(x) = \mathbf{M}^{\rightarrow}\delta_{a_1}(x) &= \sup_{h>0}(k_h * \delta_{a_1})(x) \\
&= \sup_{h>0} k_h(x - a_1) \\
&= \sup_{h>0} \begin{cases} \frac{1}{h} & \text{if } -h \le x - a_1 \le 0, \\ 0 & \text{otherwise.} \end{cases} \\
&= \begin{cases} \frac{1}{|x-a_1|} & \text{if } x \le a_1, \\ 0 & \text{if } x > a_1. \end{cases}
\end{aligned}$$

Thus, the set where $\mathbf{M}^{\rightarrow}\mu(x) > \lambda$ is given by

$$\begin{aligned}
|\{x \in \mathbb{R} : \mathbf{M}^{\rightarrow}\mu(x) > \lambda\}| &= \left|\{-\infty < x \le a_1 : \frac{1}{|x - a_1|} > \lambda\}\right| \\
&= \left|\{-\infty < x \le a_1 : \frac{-1}{\lambda} + a_1 < x < \frac{1}{\lambda} + a_1\}\right| \\
&= \frac{1}{\lambda}.
\end{aligned}$$

- **Inductive Step:** Assume that the inequality

$$\left|\{x \in \mathbb{R} : \mathbf{M}^{\rightarrow}\mu(x) > \lambda\}\right| \leq \frac{j}{\lambda}$$

holds for any choice of $\{a_k\}_{k=1}^{j}$, where $1 \leq j \leq N-1$.

• $j = N$**:** Let us prove it for $j = N$. Denote

$$E_\lambda = \{x \in \mathbb{R} : \mathbf{M}^{\rightarrow}\mu(x) > \lambda\}.$$

We may assume that $a_1 \leq a_2 \leq \cdots \leq a_N$. If $z_0 = \inf E_\lambda$, then there exists $h > 0$ such that

$$J = \#\{k : a_k \in (z_0, z_0 + h)\} \geq h\lambda.$$

Thus, we have

$$E_\lambda \subset (z_0, z_0 + h) \cup \left\{x : \mathbf{M}^{\rightarrow} \sum_{k=J+1}^{N} \delta_{a_k}(x) > \lambda\right\}.$$

Therefore, by the inductive hypothesis, we obtain

$$|E_\lambda| \leq h + \frac{N - J}{\lambda} \leq \frac{N}{\lambda}. \qquad \square$$

Best Constant for Weak-Type Inequalities

The centered Hardy-Littlewood maximal function $\mathcal{M}f$

$$\mathcal{M}f(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy$$

satisfies the weak-type $(1,1)$ inequality; that is, there exists a constant $\mathbf{C}$ such that for any $\lambda > 0$,

$$\left|\{x \in \mathbb{R}^d : \mathcal{M}f(x) > \lambda\}\right| \leq \frac{\mathbf{C}}{\lambda} \cdot \|f\|_{\mathcal{L}^1(\mathbb{R}^d)}.$$

Using the discretization method, and assuming the equidistributed case $a_{k+1} = a_k + H$ (i.e., Dirac deltas are placed at equal distances), it can be shown that the best constant in dimension $d = 1$ is given by

$$\mathbf{C} = \frac{3}{2}.$$

Best Constant for Weak-Type Inequalities

However, it remains an open question whether the best constant can, in general, be shown to be smaller than 2.

Thank You!