Hardy Littlewood Maximal Function Theorem

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Outline

Calculus

Remember from calculus that if f is a (Riemann) integrable function on $[a, b]$ and f is continuous at $x \in (a, b)$, then

$$\lim_{h \to 0} \underbrace{\frac{1}{h} \int_x^{x+h} f(y)dy}_{\text{Average of f}} = f(x)$$

Fundamental Theorem of Calculus

In other words, if

$$F(x) := \int_a^x f(y)dy,$$

then F is differentiable with

$$F'(x) = f(x)$$

at each x where f is continuous.

General Case

Let $f \in L^1(\mathbb{R}^d)$ and f be continuous at $x \in \mathbb{R}^d$. Then

$$\lim_{r \to 0} \frac{1}{|\mathcal{B}_r(x)|} \int_{\mathcal{B}_r(x)} f(y)dy = f(x),$$

where $\mathcal{B}_r(x)$ is the ball of radius r, centered at x, and $|\mathcal{B}_r(x)|$ denotes its measure.

Proof

As f is continuous at $x \in \mathbb{R}^d$, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$|y - x| < \delta \implies |f(y) - f(x)| < \varepsilon.$$

If $r < \delta$

$$\begin{aligned}
\left| \frac{1}{|\mathcal{B}_r(x)|} \int_{\mathcal{B}_r(x)} f(y)dy - f(x) \right| &= \left| \frac{1}{|\mathcal{B}_r(x)|} \int_{\mathcal{B}_r(x)} \left[f(y) - f(x) \right] dy \right| \\
&\leq \frac{1}{|\mathcal{B}_r(x)|} \int_{\mathcal{B}_r(x)} |f(y) - f(x)| dy \\
&\leq \frac{1}{|\mathcal{B}_r(x)|} \int_{\mathcal{B}_r(x)} \varepsilon dy = \varepsilon \quad \square
\end{aligned}$$

Examining the proof

Observe that

- ⋆ We only require f to be locally integrable:

$$\int_B |f| < \infty \quad \text{for any ball } B \subseteq \mathbb{R}^d.$$

- ! **But,** $f \in \mathcal{L}^1(\mathbb{R}^d)$ $\left(\text{or } \mathcal{L}^1_{loc}(\mathbb{R}^d)\right)$ may be discontinuous everywhere.
- ⋆ Therefore, we require f to be continuous for **this proof**.

Question.

If $f \in \mathcal{L}^1_{loc}(\mathbb{R}^d)$ does

$$\lim_{r \to 0} \frac{1}{|\mathcal{B}_r(x)|} \int_{\mathcal{B}_r(x)} f(y)dy$$

exist for any $x \in \mathbb{R}^d$?

Definition. For $f \in \mathcal{L}^1_{loc}(\mathbb{R}^d)$, we define the ***Hardy- Littlewood (H-L) maximal function*** $\mathcal{M}f$ by

$$\mathcal{M}f(x) := \sup_{r>0} \frac{1}{|\mathcal{B}_r(x)|} \int_{\mathcal{B}_r(x)} |f(y)|dy, \qquad x \in \mathbb{R}^d.$$

- If the set
$$\left\{ \frac{1}{|\mathcal{B}_r(x)|} \int_{\mathcal{B}_r(x)} |f(y)|dy \; : \; r > 0 \right\}$$
of the averages is **not** bounded,
$$\mathcal{M}f(x) := \infty$$

Basic Properties of $\mathcal{M}f$

- $\mathcal{M}f \geq 0$
- $|f| \leq |g| \implies \mathcal{M}f \leq \mathcal{M}g$
- $\mathcal{M}(f+g) \leq \mathcal{M}f + \mathcal{M}g$ (sub-linearity)
- $\mathcal{M}c = c$ if $c \geq 0$ is a constant.

- If f is bounded, $|f(x)| \leq A$, then

$$\begin{aligned}
\mathcal{M}f(x) &= \sup_{r>0} \frac{1}{|\mathcal{B}_r(x)|} \int_{\mathcal{B}_r(x)} |f(y)|dy \\
&\leq \frac{1}{|\mathcal{B}_r(x)|} |\mathcal{B}_r(x)| A \\
&= A
\end{aligned}$$

- $\mathcal{M}f$ could be ∞ for $f \in \mathcal{L}^1_{loc}(\mathbb{R}^d)$:

$$f(x) = x^2 \in \mathcal{L}^1_{loc}(\mathbb{R}) \implies \mathcal{M}f(x) = \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} y^2 dy$$

$$= \sup_{r>0} \frac{1}{2r} \left[\frac{(x+r)^3}{3} - \frac{(x-r)^3}{3} \right]$$

$$= \sup_{r>0} x + \frac{1}{3} r^2 = \infty$$

$\mathcal{M}f \notin \mathcal{L}^1(\mathbb{R})$

- $\mathcal{M}f \notin \mathcal{L}^1(\mathbb{R})$ for some $f \in \mathcal{L}^1(\mathbb{R})$:

$$\begin{aligned} f(x) = \chi_{[0,1]}(x) \in \mathcal{L}^1(\mathbb{R}) \implies \\ \mathcal{M}f(x) &= \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} \chi_{[0,1]}(y)dy \\ &\geq \frac{1}{2x} \int_0^{2x} \chi_{[0,1]}(y)dy \\ &= \frac{1}{2x} \int_0^1 1dy = \frac{1}{2x}, \qquad x \geq 1. \end{aligned}$$

Hence,

$$\int_R |\mathcal{M}f(x)|dx \geq \int_{\mathbb{R}} \frac{1}{2x} dx = \infty.$$

Theorem (Hardy-Littlewood Maximal Function, (1930s))

There exists a constant $A > 0$ *such that for any* $f \in \mathcal{L}^1(\mathbb{R}^d)$ *and* $\alpha > 0$,

$$\left|\left\{x \in \mathbb{R}^d \ : \ \mathcal{M}f(x) > \alpha\right\}\right| \leq \frac{A}{\alpha}||f||_{\mathcal{L}^1}$$

- We've seen that $\mathcal{M}f$ could be ∞ for $f \in \mathcal{L}^1_{loc}(\mathbb{R}^d)$.
- **However**, by the theorem, $\mathcal{M}f$ is finite *a.e.* for $f \in \mathcal{L}^1(\mathbb{R}^d)$.

Lemma (Vitalli Covering Lemma)

Let $\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_N$ be a finite collection of balls in $\mathbb{R}^d$. Then, there exists a disjoint subcollection $\mathcal{B}_{i_1}, \mathcal{B}_{i_2}, \ldots, \mathcal{B}_{i_k}$ such that

$$\sum_{j=1}^{k} |\mathcal{B}_{i_j}| \geq \frac{1}{3^d} \left| \bigcup_{j=1}^{N} \mathcal{B}_j \right|$$

Proof of the *H-L* maximal function theorem

Proof.

- Let $K \subset \{x \in \mathbb{R}^d \ : \ \mathcal{M}f(x) > \alpha\}$ be a compact subset. First, we will show that

$$|K| \leq \frac{3^d}{\alpha}||f||_{\mathcal{L}^1}.$$

Observe that

$$x \in K \implies \sup_{r>0} \frac{1}{|\mathcal{B}_r(x)|} \int_{\mathcal{B}_r(x)} |f(y)|dy > \alpha.$$

This means that there exists a ball $\mathcal{B}(x)$, centered at x, such that

$$\frac{1}{|\mathcal{B}(x)|} \int_{\mathcal{B}(x)} |f(y)|dy > \alpha.$$

Thus, $\{\mathcal{B}(x) : x \in K\}$ forms an open cover for K. As K is compact, there exists a finite subcover

$$\{\mathcal{B}_1, \mathcal{B}_2, \ldots, \mathcal{B}_N\} \subset \{\mathcal{B}(x) : x \in K\}$$

such that

$$K \subset \bigcup_{j=1}^{N} \mathcal{B}_j.$$

By the *Vitalli Covering Lemma*, there exists a disjoint subcollection $\{\mathcal{B}_{i_1}, \mathcal{B}_{i_2}, \ldots, \mathcal{B}_{i_k}\}$ such that

$$|K| \leq \left| \bigcup_{j=1}^{N} \mathcal{B}_j \right| \leq 3^d \sum_{\ell=1}^{k} |\mathcal{B}_{i_\ell}|.$$

For each $\mathcal{B}_{i_\ell}$, we have

$$\frac{1}{|\mathcal{B}_{i_\ell}|}\int_{\mathcal{B}_{i_\ell}} |f(y)|dy > \alpha \implies |\mathcal{B}_{i_\ell}| < \frac{1}{\alpha}\int_{\mathcal{B}_{i_\ell}} |f(y)|dy.$$

Hence,

$$\begin{aligned} |K| \leq 3^d \sum_{\ell=1}^{k} |\mathcal{B}_{i_\ell}| &\leq \sum_{\ell=1}^{k} \frac{1}{\alpha}\int_{\mathcal{B}_{i_\ell}} |f(y)|dy \\ &= \frac{3^d}{\alpha}\int_{\bigcup \mathcal{B}_{i_\ell}} |f(y)|dy \\ &\leq \frac{3^d}{\alpha}\int_{\mathbb{R}^d} |f(y)|dy \\ &= \frac{3^d}{\alpha}||f||_{\mathcal{L}^1}. \end{aligned}$$

• We know from the fact that for a measurable set E and for any $\varepsilon > 0$, there exists a compact subset $F \subseteq E$ such that

$$|E \setminus F| < \varepsilon.$$

As a result, we conclude that

$$\left|\left\{x \in \mathbb{R}^d \ : \ \mathcal{M}f(x) > \alpha\right\}\right| \leq \frac{A}{\alpha}||f||_{\mathcal{L}^1}. \quad \square$$

Question.

If $f \in \mathcal{L}^1_{loc}(\mathbb{R}^d)$ does

$$\lim_{r \to 0} \frac{1}{|\mathcal{B}_r(x)|} \int_{\mathcal{B}_r(x)} f(y) dy$$

exist for any $x \in \mathbb{R}^d$?

Theorem (Lebesgue Differentiation Theorem)

If $f \in \mathcal{L}^1_{loc}(\mathbb{R}^d)$, *then*

$$\lim_{r \to 0} \frac{1}{|\mathcal{B}_r(x)|} \int_{\mathcal{B}_r(x)} f(y)dy = f(x)$$

for ***a.e.*** $x \in \mathbb{R}^d$.

Proof. Let

$$\mathcal{F} = \left\{ x \in \mathbb{R}^d \ : \lim_{r\to 0} \frac{1}{|\mathcal{B}_r(x)|} \int_{\mathcal{B}_r(x)} f \neq f(x) \quad \textbf{or} \quad \text{the limit does not exist.} \right\}$$

Aim: —$\mathcal{F}$— = 0.

We will write

$$\mathcal{I}_r f(x) = \frac{1}{|\mathcal{B}_r(x)|} \int_{\mathcal{B}_r(x)} f(y) dy.$$

Note that

$$\mathcal{I}_r f(x) \to f(x) \iff \limsup_{r\to 0} |\mathcal{I}_r f(x) - f(x)| = 0.$$

Hence,

$$\mathcal{F} = \left\{ x \in \mathbb{R}^d : \limsup_{r\to 0} |\mathcal{I}_r f(x) - f(x)| > 0 \right\}.$$

Let

$$\mathcal{F}_\alpha = \left\{ x \in \mathbb{R}^d : \limsup_{r\to 0} |\mathcal{I}_r f(x) - f(x)| > \alpha \right\},$$

where $\alpha > 0$. Notice that since

$$\mathcal{F} = \bigcup_{n\geq 1} \mathcal{F}_{1/n},$$

Aim: $|F_\alpha| = 0 \; \forall \alpha > 0$.
Fix $\alpha > 0$ and let $\varepsilon > 0$ be given. Choose $g \in \mathcal{C}_c(\mathbb{R}^d)$ such that

$$||f - g||_{\mathcal{L}^1} < \varepsilon.$$

As $g \in \mathcal{C}_c(\mathbb{R}^d)$, we know that

$$\lim_{r\to 0} \mathcal{I}_r g(x) = g(x).$$

$$
\begin{aligned}
&\limsup_{r\to 0} \left|\mathcal{I}_r f(x) - f(x)\right| \\
&= \limsup_{r\to 0} \left|\mathcal{I}_r f(x) - \mathcal{I}_r g(x) + \mathcal{I}_r g(x) - g(x) + g(x) - f(x)\right| \\
&= \limsup_{r\to 0} \left|\mathcal{I}_r (f-g)(x) + \mathcal{I}_r g(x) - g(x) + g(x) - f(x)\right| \\
&\le \limsup_{r\to 0} \left[\left|\mathcal{I}_r (f-g)(x)\right| + \left|\mathcal{I}_r g(x) - g(x)\right| + \left|g(x) - f(x)\right|\right] \\
&= \limsup_{r\to 0} \left|\mathcal{I}_r (f-g)(x)\right| + \left|g(x) - f(x)\right| \\
&\le \mathcal{M}(f-g)(x) + \left|g(x) - f(x)\right|
\end{aligned}
$$

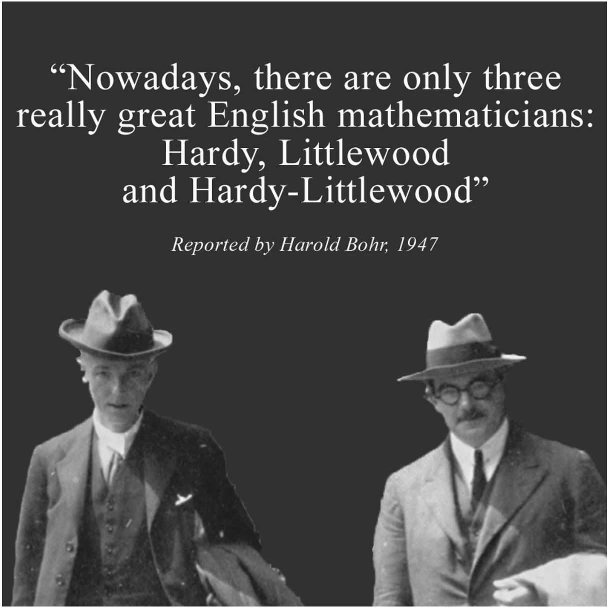
Thus,

$$
\begin{aligned}
|F_\alpha| &\le \left|\left\{x : \mathcal{M}(f-g)(x) > \frac{1}{2}\alpha\right\}\right| + \left|\left\{x : |g(x) - f(x)| > \frac{1}{2}\alpha\right\}\right| \\
&\le \frac{A}{\alpha/2}||f-g||_{\mathcal{L}^1} + \frac{1}{\alpha/2}||f-g||_{\mathcal{L}^1}.
\end{aligned}
$$

Therefore,

$$|F_\alpha| \leq \frac{2(A+1)}{\alpha}||f-g||_{\mathcal{L}^1}$$
$$< \frac{2(A+1)}{\alpha}\varepsilon.$$

As $\varepsilon > 0$ is arbitrary, $|F_\alpha| = 0$. $\square$


"Nowadays, there are only three really great English mathematicians: Hardy, Littlewood and Hardy-Littlewood"
Reported by Harold Bohr, 1947

Thank You