# Lecture Notes: The Formal Definition of a Limit

Hikmet Burak Özcan

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## 1 Introduction

The intuitive notion that a function $f(x)$ approaches a limit $L$ as $x$ approaches $c$ is formalized using the $\varepsilon$-$\delta$ definition. This definition provides a rigorous method to prove limit existence by ensuring that we can make the output $f(x)$ arbitrarily close to $L$ by restricting the input $x$ sufficiently close to $c$.

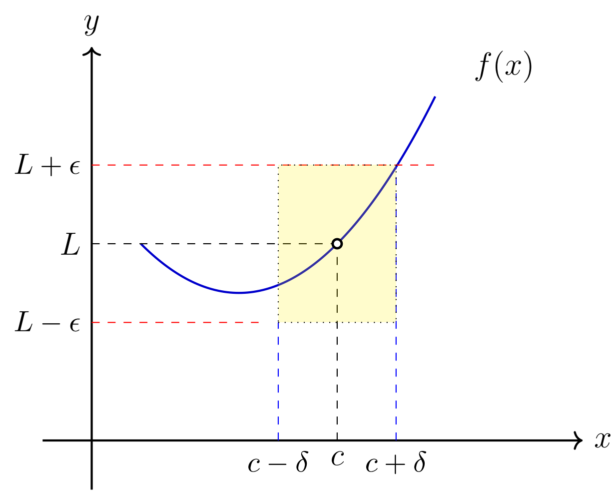


Figure 1: Geometric representation of the $\epsilon - \delta$ definition. We seek a $\delta$-window around $c$ that maps entirely into the $\epsilon$-target around $L$.

**Definition.** We say that $\lim_{x\to c} f(x) = L$ if for any $\epsilon > 0$, there exists a number $\delta > 0$ (dependent on $\epsilon$) such that:

$$0 < |x - c| < \delta \implies |f(x) - L| < \epsilon$$

**Example 1.** Let $f(x) = 2x + 3$. Show that $\lim_{x \to 2} f(x) = 7$.

**Analysis (Finding $\delta$):** We want to find $\delta > 0$ such that

$$0 < |x - 2| < \delta \implies |(2x + 3) - 7| < \epsilon.$$

$$\begin{aligned} |f(x) - 7| &= |(2x + 3) - 7| \\ &= |2x - 4| \\ &= 2|x - 2| \end{aligned}$$

We require $2|x - 2| < \epsilon$, which implies $|x - 2| < \frac{\epsilon}{2}$. Thus, we should choose $\delta = \frac{\epsilon}{2}$.

*Proof.* Let $\epsilon > 0$ be given. Choose $\delta = \frac{\epsilon}{2}$. Assume $0 < |x - 2| < \delta$. Then:

$$|f(x) - 7| = |2x - 4| = 2|x - 2| < 2\left(\frac{\epsilon}{2}\right) = \epsilon.$$

Therefore, $\lim_{x \to 2}(2x + 3) = 7$. □

**Example 2.** Let $f(x) = x^2 - x - 2$. Show that $\lim_{x \to 1} f(x) = -2$.

**Analysis:** We examine the difference $|f(x) - L|$:

$$|f(x) - (-2)| = |x^2 - x - 2 + 2| = |x^2 - x| = |x||x - 1|.$$

We need to bound the term $|x|$ by restricting $x$ near 1. Assume $|x - 1| < 1$.

$$|x - 1| < 1 \implies -1 < x - 1 < 1 \implies 0 < x < 2 \implies |x| < 2.$$

Thus, if we ensure $\delta \leq 1$, we can replace $|x|$ with 2.

*Proof.* Let $\epsilon > 0$ be given. Choose $\delta = \min\left\{1, \frac{\epsilon}{2}\right\}$. Assume $0 < |x - 1| < \delta$.

- Since $\delta \leq 1$, we have $|x - 1| < 1$, which implies $|x| < 2$.
- Since $\delta \leq \frac{\epsilon}{2}$, we have $|x - 1| < \frac{\epsilon}{2}$.

Substituting these bounds:

$$|f(x) - (-2)| = |x||x - 1| < 2 \cdot \frac{\epsilon}{2} = \epsilon.$$

Thus, the limit is $-2$. □

**Example 3.** Let $f(x) = \sqrt{x^2+3}$. Show that $\lim_{x\to 1} \sqrt{x^2+3} = 2$.

**Analysis:** We rationalize the expression to isolate $|x-1|$:

$$\begin{aligned} |f(x)-2| &= |\sqrt{x^2+3}-2| \\ &= \left| \frac{(\sqrt{x^2+3}-2)(\sqrt{x^2+3}+2)}{\sqrt{x^2+3}+2} \right| \\ &= \left| \frac{x^2+3-4}{\sqrt{x^2+3}+2} \right| \\ &= \frac{|x^2-1|}{\sqrt{x^2+3}+2} \\ &= |x-1| \cdot \left| \frac{x+1}{\sqrt{x^2+3}+2} \right| \end{aligned}$$

We need to bound $K(x) = \left| \frac{x+1}{\sqrt{x^2+3}+2} \right|$. Assume $|x-1| < 1$ (so $0 < x < 2$).

- **Numerator:** $|x+1| < 3$ (since $x < 2$).
- **Denominator:** $x^2 > 0 \implies \sqrt{x^2+3} > \sqrt{3}$. Thus, the denominator is greater than $\sqrt{3}+2$.

Combining these:

$$\left| \frac{x+1}{\sqrt{x^2+3}+2} \right| < \frac{3}{\sqrt{3}+2}.$$

*Proof.* Let $\epsilon > 0$ be given. Choose $\delta = \min\left\{1, \frac{\sqrt{3}+2}{3}\epsilon\right\}$. Assume $0 < |x-1| < \delta$. Since $\delta \le 1$, our bound holds: $\left| \frac{x+1}{\sqrt{x^2+3}+2} \right| < \frac{3}{\sqrt{3}+2}$. Then:

$$|f(x)-2| = |x-1| \left| \frac{x+1}{\sqrt{x^2+3}+2} \right| < \delta \cdot \frac{3}{\sqrt{3}+2} \le \left( \frac{\sqrt{3}+2}{3}\epsilon \right) \frac{3}{\sqrt{3}+2} = \epsilon.$$

Therefore, the limit is 2. □

**Problem 1.** Prove that $\lim_{x \to 3}(x^2 - 2x) = 3$.

**Problem 2.** Prove that $\lim_{x \to 2} \frac{1}{x} = \frac{1}{2}$.