MATH 145 Calculus for Engineering and Science I
Midterm 1

November 3th, 2025

1. For which numbers a, b, c, and d will the function

$$f(x) = \frac{ax+b}{cx+d}$$

satisfy $f(f(x)) = x$ for all x?

$$f\big(f(x)\big) = x \Leftrightarrow f^{-1}(x) = f(x)$$

$$f(x) = \frac{ax+b}{cx+d} \Rightarrow f^{-1}(x) = \frac{b-dx}{cx-a}$$

$$\frac{ax+b}{cx+d} = \frac{b-dx}{cx-a} \Rightarrow (ax+b)(cx-a) = (cx+d)(b-dx)$$

$$\Rightarrow acx^2 - a^2x + \cancel{bc}x - ab = \cancel{bc}x - cdx^2 + bd - d^2x$$

$$\Rightarrow (ac+cd)x^2 + (d^2-a^2)x + (-ab-bd) = 0$$

$$\left.\begin{aligned} c(a+d) &= 0 \\ b(a+d) &= 0 \\ d^2-a^2 &= 0 \end{aligned}\right\} \Rightarrow a = -d$$

2. i. Show that the straight line through (a, b) with slope m is the graph of the function $f(x) = m(x - a) + b$; it is immediately clear from the point-slope form that the slope is m, and that the value of f at a is b.

ii. For $a \neq c$, show that the straight line through (a, b) and (c, d) is the graph of the function

$$f(x) = \frac{d - b}{c - a}(x - a) + b\,.$$

i) The point-slope form of a linear equation passing through the point (x_1, y_1) with slope m is

$$y - y_1 = m(x - x_1)$$

For the given point (a, b),

$$y - b = m(x - a) \Rightarrow y = m(x-a) + b$$

Since the graph of $f(x)$ is defined by $y = f(x)$, we have

$$f(x) = m(x-a) + b, \qquad f(a) = b.$$

ii) Slope is $m = \dfrac{d-b}{c-a}$, $c \neq a$

By i), $f(x) = \dfrac{d-b}{c-a}(x-a) + b$

3. i. Let R_θ be rotation by angle of θ. Show that $R_\theta(v) \cdot R_\theta(w) = v \cdot w$ ($\cdot$ is the dot (or scalar) product for vectors).

ii. Let $e = (1, 0)$ be the vector of length 1 pointing along the first axis, and let $w = (\cos\theta, \sin\theta)$; this is a vector of length 1 that makes an angle of with the first axis. Calculate that

$$e \cdot w = \cos\theta\,.$$

i) $v \cdot w = |v||w|\cos\alpha$, where α is the angle between u & v.

Since a rotation R_θ is a rigit transformation, it does not change the magnitude of a vector. So,

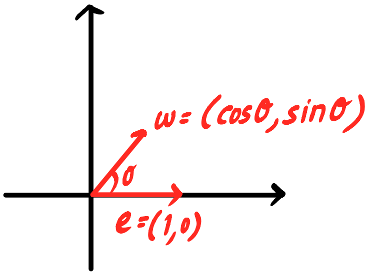
$$|R_\theta(v)| = |v| \quad \& \quad |R_\theta(w)| = |w|$$

Also, a rotation rotates both vectors by the same angle θ, preserving the angle α between them. So,

$$\begin{aligned} R_\theta(v) \cdot R_\theta(w) &= |R_\theta(v)| \cdot |R_\theta(w)| \cdot \cos\alpha \\ &= |v| \cdot |w| \cdot \cos\alpha \\ &= v \cdot w \end{aligned}$$

ii) Using i)

$$\begin{aligned} e \cdot w &= |e| \cdot |w| \cdot \cos\theta \\ &= 1 \cdot 1 \cdot \cos\theta \\ &= \cos\theta\,. \end{aligned}$$



4. Prove that $\lim_{x\to a} f(x)$ exists if $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$.

Assume

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = \mathcal{L}$$

Then, for any $\varepsilon > 0$

i) $\exists \delta_1 > 0 : \quad a < x < a + \delta_1 \Rightarrow |f(x) - \mathcal{L}| < \varepsilon$

ii) $\exists \delta_2 > 0 : \quad a - \delta_2 < x < a \Rightarrow |f(x) - \mathcal{L}| < \varepsilon$

Let $\delta = \min\{\delta_1, \delta_2\}$. Combining i) & ii)

$$a - \delta < x < a + \delta \Rightarrow |f(x) - \mathcal{L}| < \varepsilon, \text{ i.e.}$$

$$0 < |x - a| < \delta \Rightarrow |f(x) - \mathcal{L}| < \varepsilon$$

So, $\lim_{x\to a} f(x) = \mathcal{L}$.

5. Suppose that f is a function satisfying $|f(x)| \leq |x|$ for all x. Show that f is continuous at 0. (Notice that $f(0)$ must equal 0.)

Suppose $|f(x)| \leqslant |x|$ for all x.
In particular, $|f(0)| \leqslant 0 \Rightarrow f(0) = 0$.

$|f(x)| \leqslant |x| \Rightarrow -|x| \leqslant f(x) \leqslant |x|$, where

$\lim_{x \to 0} -|x| = \lim_{x \to 0} |x| = 0$.

By the Sandwich theorem, $\lim_{x \to 0} f(x) = 0$

Since $f(0) = 0$, we conclude $\lim_{x \to 0} f(x) = f(0)$.

So, f is continuous at $x = 0$.