MATH 145 Calculus for Engineering and Science I

Midterm 2

- KEY -

December 8th, 2025

1. Find the derivative of the following function

$$F(x) = \int_x^b \frac{1}{1+t^2+\sin^2 t} dt$$

$$F(x) = -\int_b^x \frac{1}{1+t^2+\sin t} dt \; ,$$

$f(t) = \dfrac{1}{1+t^2+\sin t}$ is continuous on $[b,c]$ for any $c \in \mathbb{R}$.

By F.T.C.,

$$\frac{d}{dx} F(x) = -\frac{1}{1+x^2+\sin x}$$

2. Suppose that f is differentiable with derivative $f'(x) = (1+x^3)^{-1/2}$. Show that that $g = f^{-1}$ satisfies $g''(x) = \frac{3}{2}g(x)^2$.

Let $g = f^{-1}$. Then $f \circ g(x) = x$ for any x.

By the chain rule,

$$f'(g(x)) \cdot g'(x) = 1 \Rightarrow g'(x) = \frac{1}{f'(g(x))}.$$

Substitute $g(x)$ into $f'(x) = (1+x^3)^{-1/2}$:

$$g'(x) = \frac{1}{(1+g(x)^3)^{-1/2}} = (1+g(x)^3)^{1/2}$$

Then

$$g''(x) = \frac{3g(x)^2 \cdot g'(x)}{2(1+g(x)^3)^{1/2}}$$

$$= \frac{3g(x)^2 \cdot \cancel{(1+g(x)^3)^{1/2}}}{2\cancel{(1+g(x)^3)^{1/2}}}$$

$$= \frac{3}{2}g(x)^2.$$

3. Show that $f(x) = \frac{ax+b}{cx+d}$ is one-to-one if $ad - bc \neq 0$, and find f^{-1}.

Assume $f(x_1) = f(x_2)$. We must show $x_1 = x_2$.

Then $$\frac{ax_1+b}{cx_1+d} = \frac{ax_2+b}{cx_2+d}$$

$$\Rightarrow (ax_1+b)(cx_2+d) = (ax_2+b)(cx_1+d)$$
$$\Rightarrow \cancel{acx_1x_2} + adx_1 + bcx_2 + \cancel{bd} = \cancel{acx_1x_2} + adx_2 + bcx_1 + \cancel{bd}$$

$$\Rightarrow (ad-bc)x_1 = (ad-bc)x_2$$

$$\Rightarrow x_1 = x_2 \quad \text{if } ad - bc \neq 0.$$

Set $y = \frac{ax+b}{cx+d}$ and solve for x:

$$y(cx+d) = ax+b \Rightarrow cxy + dy = ax+b$$
$$\Rightarrow cxy - ax = b - dy$$
$$\Rightarrow x(cy-a) = b-dy$$
$$\Rightarrow x = \frac{b-dy}{cy-a}$$

So, $f^{-1}(x) = \frac{-dx+b}{cx+a}$.

4. Prove that

$$\int_{ca}^{cb} f(t)dt = c\int_a^b f(ct)dt$$

(Hint: If $P = \{t_0, \cdots, t_n\}$ and $P' = \{ct_0, \cdots, ct_n\}$ of $[a, b]$, then $m_i = \inf\{f(ct) : t_{i-1} \leq t \leq t_i\} = \inf\{f(t) : ct_{i-1} \leq t \leq ct_i\}$.)

Let $P = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[a,b]$ and let $P' = \{ct_0, ct_1, \ldots, ct_n\}$ be a corresponding partition of $[ca, cb]$.

The lower Riemann sum for $\int_{ca}^{cb} f(x)dx$ is

$$\mathcal{L}(f, P') = \sum_{i=1}^{n} m_i' \Delta x_i', \quad \begin{aligned} \Delta x_i' &= ct_i - ct_{i-1} \\ &= c(t_i - t_{i-1}) \\ &= c\,\Delta t_i \end{aligned}$$

$$m_i' = \inf\{f(x) : ct_{i-1} \leq x \leq ct_i\}$$

Let $g(t) := f(ct)$

The lower Riemann sum for $c\int_a^b f(ct)dt$ is

$$\mathcal{L}(g, P) = c\sum_{i=1}^{n} m_i \Delta t_i, \quad m_i = \inf\{f(ct) : t_{i-1} \leq t \leq t_i\}$$

From the hint, $m_i = m_i'$.

Substitute into the sum:

$$\mathcal{L}(f, P') = \sum_{i=1}^{n} m_i (c\Delta t_i) = c\sum_{i=1}^{n} m_i \Delta t_i = c\mathcal{L}(g, P)$$

$$\Rightarrow \int_{ca}^{cb} f(x)dx = c\int_a^b f(ct)dt$$

II. WAY

Let $u = ct$. Then $du = c\,dt \Rightarrow$

So, $c\int_a^b f(ct)\,dt = c\int_{ca}^{cb} \frac{f(u)}{c}\,du = \int_{ca}^{cb} f(u)\,du$ ∎

5. Find $(f^{-1})'(0)$ if

i. $f(x) = \int_0^x (1+\sin(\sin t))dt$

ii. $f(x) = \int_1^x \cos(\cos t)dt$

Note that

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

i) First, let us find x_0 with $f(x_0) = 0$.

$$f(x_0) = \int_0^{x_0} (1+\sin(\sin t))dt = 0 \Rightarrow x_0 = 0$$

$\Rightarrow f^{-1}(0) = 0$.

Then $(f^{-1})'(0) = \dfrac{1}{f'(0)}$,

where, by F.T.C., $f'(x) = 1 + \sin(\sin x)$

So, $(f^{-1})'(0) = \dfrac{1}{1+\sin(\sin 0)} = 1$.

ii) $f'(x) = \cos(\cos x)$

$$\Rightarrow (f^{-1})'(0) = \frac{1}{\cos(\cos 0)} = \frac{1}{\cos 1}.$$