MATH 145 Calculus for Engineering and Science I
Recitation 2
Solution Key

October 20th, 2025

1. Find the domain of the functions defined by the following formulas.
   (a) $\sqrt{1-\sqrt{1-x^2}}$
   (b) $\frac{x+1}{x-1}+\frac{x-1}{x-2}$
   (c) $\sqrt{1-x+\sqrt{x-2}}$

   Solution:

   (a) We need $1-x^2 \geq 0$ and $1-\sqrt{1-x^2} \geq 0$.
   $$1-x^2 \geq 0 \implies 1 \geq x^2 \implies -1 \leq x \leq 1,$$
   and
   $$1-\sqrt{1-x^2} \geq 0 \implies 1 \geq \sqrt{1-x^2} \implies 1^2 \geq (\sqrt{1-x^2})^2 \implies 1 \geq 1-x^2 \implies x^2 \geq 0.$$
   Combining two conditions, we get the domain is $[-1,1]$.

   (b) Since the function is undefined when either denominator is zero, we require
      i. $x-1 \neq 0 \implies \mathbf{x \neq 1}$.
      ii. $x-2 \neq 0 \implies \mathbf{x \neq 2}$.

   So, the domain is $\mathbb{R} \setminus \{1,2\}$, or $(-\infty,1) \cup (1,2) \cup (2,\infty)$.

   (c) We need to ensure all arguments of square roots are non-negative.
      i. $x-2 \geq 0 \implies x \geq 2$.
      ii. $1-x+\sqrt{x-2} \geq 0 \implies \sqrt{x-2} \geq x-1$.

   From condition (i), we know $x \geq 2$. This means $x-1$ is positive, specifically $x-1 \geq 1$. Since both sides are non-negative, we can square them
   $$(\sqrt{x-2})^2 \geq (x-1)^2 \implies x-2 \geq x^2-2x+1 \implies 0 \geq x^2-3x+3.$$
   To solve this, we find the roots of $y = x^2-3x+3$. The discriminant is
   $$\Delta = b^2-4ac = (-3)^2-4(1)(3) = 9-12 = -3.$$
   Since $\Delta < 0$ and the leading coefficient ($a=1$) is positive, the parabola $y = x^2-3x+3$ is always positive. It is never less than or equal to 0. There are no x-values that satisfy the condition (ii).

   The domain is the empty set $\emptyset$.

2. A function f is called **even** if $f(x) = f(-x)$ and **odd** if $f(x) = -f(-x)$.
   For example:
   $$f(x) = x^2, \quad f(x) = |x|, \quad f(x) = \cos x$$
   are even functions, while
   $$f(x) = x, \quad f(x) = \sin x$$
   are odd functions.

(a) Determine whether $f+g$ is even, odd, or not necessarily either, in the four cases obtained by choosing f even or odd, and g even or odd.

(b) Do the same for the product $f \cdot g$.

(c) Do the same for the composition $f \circ g$.

Solution:

(a) **Sum** $f+g$

- **Even + Even:**

$$(f+g)(-x) = f(-x) + g(-x) = f(x) + g(x) = (f+g)(x),$$

so $f+g$ is **even**.

- **Odd + Odd:**

$$(f+g)(-x) = f(-x) + g(-x) = -f(x) - g(x) = -(f+g)(x),$$

so $f+g$ is **odd**.

- **Even + Odd (or Odd + Even):**

$$(f+g)(-x) = f(-x) + g(-x) = f(x) - g(x) \neq \pm(f(x) + g(x)),$$

so in general $f+g$ is **neither even nor odd**.

(b) **Product** $f \cdot g$

- **Even $\cdot$ Even:**

$$(fg)(-x) = f(-x)g(-x) = f(x)g(x) = (fg)(x),$$

so fg is **even**.

- **Odd $\cdot$ Odd:**

$$(fg)(-x) = f(-x)g(-x) = (-f(x))(-g(x)) = f(x)g(x) = (fg)(x),$$

so fg is **even**.

- **Even $\cdot$ Odd (or Odd $\cdot$ Even):**

$$(fg)(-x) = f(-x)g(-x) = f(x)(-g(x)) = -(f(x)g(x)) = -(fg)(x),$$

so fg is **odd**.

(c) **Composition** $f \circ g$

- **If g is even:** $g(-x) = g(x)$, so

$$(f \circ g)(-x) = f(g(-x)) = f(g(x)) = (f \circ g)(x),$$

and therefore $f \circ g$ is **even** for any f.

- **If g is odd:** $g(-x) = -g(x)$. Then

$$(f \circ g)(-x) = f(g(-x)) = f(-g(x)).$$

- **If f is even**, $f(-u) = f(u)$, hence $f(-g(x)) = f(g(x))$, so $f \circ g$ is **even**.
- **If f is odd**, $f(-u) = -f(u)$, hence $f(-g(x)) = -f(g(x))$, so $f \circ g$ is **odd**.

3. Draw the set of all points (x, y) satisfying each of the following conditions.

(a) $|x| + |y| = 1$

(b) $x^2 + y^2 = 0$

(c) $x^2 - 2x + y^2 = 4$

Solution: We are asked to describe and sketch the set of all points (x, y) satisfying each condition.

(a) We analyze this in four quadrants

- **Quadrant I ($x \geq 0, y \geq 0$):** $x + y = 1 \implies y = 1 - x$.
- **Quadrant II ($x \leq 0, y \geq 0$):** $-x + y = 1 \implies y = 1 + x$.
- **Quadrant III ($x \leq 0, y \leq 0$):** $-x - y = 1 \implies y = -1 - x$.
- **Quadrant IV ($x \geq 0, y \leq 0$):** $x - y = 1 \implies y = x - 1$.

These four line segments connect to form a **square (or "diamond") with vertices at** $(1, 0), (0, 1), (-1, 0), (0, -1)$.

(b) For all real numbers x and y, $x^2 \geq 0$ and $y^2 \geq 0$. The sum of two non-negative numbers can only be zero if both numbers are individually zero.

$$x^2 = 0 \implies x = 0$$

$$y^2 = 0 \implies y = 0$$

The only point satisfying this condition is the **origin** $(0, 0)$.

(c) We complete the square for the x-terms.

$$(x^2 - 2x) + y^2 = 4$$

$$(x^2 - 2x + 1) + y^2 = 4 + 1$$
$$(x - 1)^2 + y^2 = 5$$

This is the standard equation of a **circle centered at** $(1, 0)$ **with a radius of** $r = \sqrt{5}$.

4. Given vectors $v = (v_1, v_2)$ and $w = (w_1, w_2)$, we define the number

$$v \cdot w = v_1 w_1 + v_2 w_2,$$

which is called the **dot product** or **scalar product** of v and w.

(a) Given a vector v, find a vector w such that $v \cdot w = 0$. Describe the set of all such vectors w.

(b) Show that:

$$v \cdot w = w \cdot v,$$
$$v \cdot (w + z) = v \cdot w + v \cdot z,$$
$$a(v \cdot w) = (av) \cdot w = v \cdot (aw).$$

Solution:

(a) We need to solve $v \cdot w = v_1 w_1 + v_2 w_2 = 0$.

- **Case 1:** $v = (0, 0)$**.** The equation is $0 \cdot w_1 + 0 \cdot w_2 = 0$, which is $0 = 0$. This is true for **any vector** $w \in \mathbb{R}^2$.
- **Case 2:** $v \neq (0, 0)$**.** We can solve for one component, e.g., $v_1 w_1 = -v_2 w_2$. A non-trivial solution is $w_1 = v_2$ and $w_2 = -v_1$, because $v_1(v_2) + v_2(-v_1) = 0$. Any scalar multiple of this vector will also work: $w = t(v_2, -v_1)$ for any $t \in \mathbb{R}$.

Description: The set of all such vectors w is the line through the origin that is perpendicular (orthogonal) to the vector v.

(b) Let $v = (v_1, v_2)$, $w = (w_1, w_2)$, $z = (z_1, z_2)$, and $a \in \mathbb{R}$.

- $v \cdot w = w \cdot v$ **(Commutativity)**

$$v \cdot w = v_1 w_1 + v_2 w_2$$
$$w \cdot v = w_1 v_1 + w_2 v_2$$

Since $v_i w_i = w_i v_i$ for real numbers, $v \cdot w = w \cdot v$.

- $v \cdot (w + z) = v \cdot w + v \cdot z$ **(Distributivity)**

$$\begin{aligned}
\textbf{LHS: } v \cdot (w + z) &= (v_1, v_2) \cdot (w_1 + z_1, w_2 + z_2) \\
&= v_1(w_1 + z_1) + v_2(w_2 + z_2) \\
&= v_1w_1 + v_1z_1 + v_2w_2 + v_2z_2 \\
\textbf{RHS: } v \cdot w + v \cdot z &= (v_1w_1 + v_2w_2) + (v_1z_1 + v_2z_2) \\
&= v_1w_1 + v_1z_1 + v_2w_2 + v_2z_2
\end{aligned}$$

  LHS = RHS.

- $a(v \cdot w) = (av) \cdot w = v \cdot (aw)$ **(Scalar Associativity)**

$$\begin{aligned}
&a(v \cdot w) = a(v_1w_1 + v_2w_2) = \mathbf{av_1w_1} + \mathbf{av_2w_2} \\
&(av) \cdot w = (av_1, av_2) \cdot (w_1, w_2) = (av_1)w_1 + (av_2)w_2 = \mathbf{av_1w_1} + \mathbf{av_2w_2} \\
&v \cdot (aw) = (v_1, v_2) \cdot (aw_1, aw_2) = v_1(aw_1) + v_2(aw_2) = \mathbf{av_1w_1} + \mathbf{av_2w_2}
\end{aligned}$$

  All three expressions are equal.

5. Consider a cylinder with a generator perpendicular to the horizontal plane; the only requirement for a point (x, y, z) to lie on this cylinder is that (x, y) lies on a circle:

$$x^2 + y^2 = C^2$$

Show that the intersection of a plane with this cylinder can be described by an equation of the form

$$(\alpha x + \beta)^2 + y^2 = C^2.$$

What are the possibilities?

Solution: Consider an arbitrary plane. By rotating the coordinate system about the z-axis (which preserves the cylinder), we can simplify the equation of the plane. There are two main cases:

(a) **Non-vertical Plane:** After rotation, the plane can be written as

$$z = kx + d, \quad \text{for some } k, d \in \mathbb{R}.$$

The intersection of this plane with the cylinder consists of points (x, y, z) satisfying

$$x^2 + y^2 = C^2 \quad \text{and} \quad z = kx + d.$$

On the plane, we can use (x, y) as coordinates (since the plane is the graph of z over the (x, y)-plane). In these coordinates, the intersection curve is given by

$$x^2 + y^2 = C^2.$$

This is already of the desired form

$$(\alpha x + \beta)^2 + y^2 = C^2$$

with $\alpha = 1$ and $\beta = 0$.

(b) **Vertical Plane:** After rotation, the plane can be written as

$$x = d, \quad \text{for some } d \in \mathbb{R}.$$

The intersection with the cylinder is given by

$$x = d, \quad d^2 + y^2 = C^2.$$

On the plane, we can use coordinates (y, z). However, if we relabel the coordinates as (x, y) (where x is the coordinate along the direction of the plane's intersection with the horizontal plane, and y is the vertical coordinate), then the equation becomes

$$d^2 + y^2 = C^2.$$

This is of the form

$$(\alpha x + \beta)^2 + y^2 = C^2$$

with $\alpha = 0$ and $\beta = d$.

In both cases, the intersection curve on the plane can be described by an equation of the form

$$(\alpha x + \beta)^2 + y^2 = C^2,$$

where (x, y) are suitable coordinates on the plane.

The nature of the intersection depends on the parameters α and β:

- **If $\alpha \neq 0$:**
  Then the equation can be transformed into $X^2 + y^2 = C^2$ by setting $X = \alpha x + \beta$. This represents a circle in the (X, y) coordinates. Since these coordinates are affine (not necessarily orthonormal with respect to the Euclidean metric on the plane), the actual curve in the plane is an *ellipse* (a circle being a special case when the plane is horizontal).
- **If $\alpha = 0$:**
  Then the equation becomes $\beta^2 + y^2 = C^2$, or

  $$y^2 = C^2 - \beta^2.$$

  - If $|\beta| < C$, then $y = \pm\sqrt{C^2 - \beta^2}$. This represents *two parallel lines* (since x is free).
  - If $|\beta| = C$, then $y = 0$, so the intersection is a *single line* (a degenerate case).
  - If $|\beta| > C$, then there are no real points of intersection.