MATH 145: Calculus for Engineering and Science I
Recitation 5 Solution Key

November 17th, 2025

1. Find $f''(x)$ for the following functions:

   1. $f(x) = x^5$

      Solution: First, find the first derivative using the power rule:

      $$f'(x) = 5x^4$$

      Differentiate again to find the second derivative:

      $$f''(x) = 20x^3$$

   2. $f'(x) = x^4$

      Solution: Note that the problem gives $f'(x)$, not $f(x)$. To find $f''(x)$, we differentiate $f'(x)$ once:

      $$f''(x) = \frac{d}{dx}(x^4) = 4x^3$$

   3. $f(x+3) = x^5$

      Solution: Let $u = x + 3$, which implies $x = u - 3$. Substituting this into the function definition:

      $$f(u) = (u-3)^5$$

      Now differentiate with respect to u:

      $$f'(u) = 5(u-3)^4$$
      $$f''(u) = 20(u-3)^3$$

      Substitute $u = x + 3$ back into the expression for $f''(u)$ to find $f''(x)$. Wait, we need $f''(x)$, which is the function evaluated at x. From $f''(u) = 20(u-3)^3$, we simply replace the variable u with x:

      $$f''(x) = 20(x-3)^3$$

---

2. Prove that if $f(x) = 1/x^2$, then $f'(a) = -2/a^3$ for $a \neq 0$.

Proof. We use the limit definition of the derivative at $x = a$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

Substitute $f(x) = \frac{1}{x^2}$

$$f'(a) = \lim_{h\to 0} \frac{\frac{1}{(a+h)^2} - \frac{1}{a^2}}{h}$$

Combine the fractions in the numerator

$$f'(a) = \lim_{h\to 0} \frac{\frac{a^2-(a+h)^2}{a^2(a+h)^2}}{h} = \lim_{h\to 0} \frac{a^2 - (a^2 + 2ah + h^2)}{ha^2(a+h)^2}$$

Simplify the numerator

$$f'(a) = \lim_{h\to 0} \frac{-2ah - h^2}{ha^2(a+h)^2} = \lim_{h\to 0} \frac{h(-2a-h)}{ha^2(a+h)^2}$$

Cancel the factor of h

$$f'(a) = \lim_{h\to 0} \frac{-2a-h}{a^2(a+h)^2}$$

Evaluate the limit as $h \to 0$

$$f'(a) = \frac{-2a-0}{a^2(a+0)^2} = \frac{-2a}{a^4} = -\frac{2}{a^3}$$

□

---

3. Find $f'(0)$ if $f(x) = g(x)\sin(1/x)$ for $x \neq 0$ and $f(0) = 0$, given $g(0) = g'(0) = 0$.
   Solution: Using the definition of the derivative at $x = 0$

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{g(h)\sin(1/h) - 0}{h}$$

Rearranging terms

$$f'(0) = \lim_{h\to 0} \left[\frac{g(h)}{h} \cdot \sin\left(\frac{1}{h}\right)\right]$$

We know that $g(0) = 0$, so

$$\lim_{h\to 0} \frac{g(h)}{h} = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = g'(0) = 0$$

Since sine is bounded ($-1 \leq \sin(1/h) \leq 1$), and $\frac{g(h)}{h} \to 0$, by the Squeeze Theorem

$$f'(0) = 0 \cdot (\text{bounded}) = 0$$

Problem 4

4. The area between two varying concentric circles is always 9π m^2. The rate of change of the larger circle's area is 10π m^2/sec. How fast is the circumference of the smaller circle changing when it has area 16π m^2?

Solution: Let A_L, A_S be the areas and C_S, r_S be the circumference and radius of the smaller circle.

We are given $A_L - A_S = 9\pi$. Differentiating with respect to time t

$$\frac{dA_L}{dt} - \frac{dA_S}{dt} = 0 \implies \frac{dA_S}{dt} = \frac{dA_L}{dt} = 10\pi$$

At the instant $A_S = 16\pi$

$$\pi r_S^2 = 16\pi \implies r_S = 4 \text{ m}$$

Differentiating $A_S = \pi r_S^2$

$$\frac{dA_S}{dt} = 2\pi r_S \frac{dr_S}{dt}$$

Substituting values (10π for rate, 4 for radius)

$$10\pi = 2\pi(4)\frac{dr_S}{dt} \implies 10\pi = 8\pi\frac{dr_S}{dt} \implies \frac{dr_S}{dt} = \frac{5}{4} \text{ m/s}$$

Since $C_S = 2\pi r_S$, then $\frac{dC_S}{dt} = 2\pi\frac{dr_S}{dt}$

$$\frac{dC_S}{dt} = 2\pi\left(\frac{5}{4}\right) = \frac{5\pi}{2} \text{ m/s}$$