MATH 145: Calculus for Engineering and Science I
Recitation 6 Solution Key

November 24th, 2025

1. Find the maximum and minimum values on the indicated intervals.

   1. $f(x) = x^3 - x^2 - 8x + 1$ on $[-2, 2]$

      Solution: Find the derivative to locate critical points

      $$f'(x) = 3x^2 - 2x - 8$$

      Set $f'(x) = 0$

      $$(3x+4)(x-2) = 0 \implies x = -\frac{4}{3}, \quad x = 2$$

      Both critical points are in $[-2, 2]$. We evaluate $f(x)$ at the endpoints and critical points

      $$\begin{aligned} f(-2) &= -8 - 4 + 16 + 1 = 5 \\ f(-4/3) &= \left(-\frac{4}{3}\right)^3 - \left(-\frac{4}{3}\right)^2 - 8\left(-\frac{4}{3}\right) + 1 = \frac{203}{27} \approx 7.52 \\ f(2) &= 8 - 4 - 16 + 1 = -11 \end{aligned}$$

      Hence, max $= \frac{203}{27}$, min $= -11$.

   2. $f(x) = \frac{1}{x^5+x+1}$ on $[-1/2, 1]$

      Solution: Differentiate using the chain rule

      $$f'(x) = -\frac{5x^4+1}{(x^5+x+1)^2}$$

      Since $x^4 \geq 0$, the numerator is always positive. The denominator is always positive. Thus, $f'(x) < 0$ for all x. The function is strictly decreasing.

      - Max at left endpoint: $f(-1/2) = \frac{1}{-1/32-16/32+1} = \frac{32}{15}$.
      - Min at right endpoint: $f(1) = \frac{1}{3}$.

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2. Find the following limits

   1. $\lim_{x\to 0} \frac{x}{\tan x}$

      Solution:

      $$\lim_{x\to 0} \frac{x}{\frac{\sin x}{\cos x}} = \lim_{x\to 0} \left(\frac{x}{\sin x} \cdot \cos x\right) = 1 \cdot 1 = 1$$

2. $\lim_{x\to 0} \frac{\cos^2 x - 1}{x^2}$

   Solution: Using the identity $\cos^2 x - 1 = -\sin^2 x$:

$$\lim_{x\to 0} \frac{-\sin^2 x}{x^2} = -\left(\lim_{x\to 0} \frac{\sin x}{x}\right)^2 = -(1)^2 = -1$$

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3. Find $f'(0)$ if $f(x) = g(x)/x$ for $x \neq 0$, $f(0) = 0$, with $g(0) = g'(0) = 0, g''(0) = 17$.
   Solution: Using the definition of the derivative

$$f'(0) = \lim_{h\to 0} \frac{f(h) - f(0)}{h} = \lim_{h\to 0} \frac{g(h)/h}{h} = \lim_{h\to 0} \frac{g(h)}{h^2}$$

This is a 0/0 form. Applying L'Hôpital's Rule twice

$$\lim_{h\to 0} \frac{g'(h)}{2h} \xrightarrow{L'H} \lim_{h\to 0} \frac{g''(h)}{2} = \frac{17}{2}$$

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4. Let $f(x) = x^4 \sin^2(1/x)$ for $x \neq 0$ and $f(0) = 0$.

   1. Prove that 0 is a local minimum point for f.

      Proof: For all $x \neq 0$, $x^4 > 0$ and $\sin^2(1/x) \geq 0$. Thus $f(x) \geq 0$. Since $f(0) = 0$, $f(x) \geq f(0)$ for all x. Therefore, 0 is a global minimum.

   2. Prove $f'(0) = f''(0) = 0$.

      Proof:

$$f'(0) = \lim_{h\to 0} \frac{h^4 \sin^2(1/h)}{h} = \lim_{h\to 0} h^3 \sin^2(1/h)$$

      Since $0 \leq |\sin^2(1/h)| \leq 1$, by Squeeze Theorem, the limit is 0.

      First, find $f'(x)$ for $x \neq 0$.

$$f'(x) = 4x^3 \sin^2(1/x) - 2x^2 \sin(1/x)\cos(1/x)$$

      Now apply the limit definition for $f''(0)$

$$f''(0) = \lim_{h\to 0} \frac{4h^3 \sin^2(1/h) - 2h^2 \sin(1/h)\cos(1/h)}{h}$$

$$= \lim_{h\to 0} \left(4h^2 \sin^2(1/h) - 2h \sin(1/h)\cos(1/h)\right)$$

      Both terms go to 0 as $h \to 0$ (Squeeze Theorem), so $f''(0) = 0$.