MATH 145: Calculus for Engineering and Science I
Recitation 8
Solution Key

December 15th, 2025

Problems

1. Let $f(x) = \frac{\sin x}{x}$ for $x \neq 0$ and $f(0) = 1$. Find $f'(0)$ and $f''(0)$.

2. Show that for x, y such that $x + y \neq k\pi + \pi/2$:

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

3. Show that if $n, m \in \mathbb{N}$, then $\int_{-\pi}^{\pi} \sin(mx)\cos(nx)\,dx = 0$.

4. Find the following limits by L'Hopital's Rule:

(i) $\lim_{x \to 0} \frac{e^x - 1 - x - x^2}{x^3}$

(ii) $\lim_{x \to \frac{\pi}{4}} (\tan x)^{\tan 2x}$

5. Show that $e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$.

6. Show that if $f(x) = \int_0^x f(t)dt$, then $f = 0$.

Solutions

1. Derivatives of the Sinc Function

(i) Find $f'(0)$: Using the limit definition and L'Hopital's Rule:

$$f'(0) = \lim_{x \to 0} \frac{f(x) - 1}{x} = \lim_{x \to 0} \frac{\frac{\sin x}{x} - 1}{x} = \lim_{x \to 0} \frac{\sin x - x}{x^2}$$

Applying L'Hopital's Rule repeatedly:

$$= \lim_{x \to 0} \frac{\cos x - 1}{2x} = \lim_{x \to 0} \frac{-\sin x}{2} = \boxed{0}$$

(ii) Find $f''(0)$: Using the limit definition $f''(0) = \lim_{x\to 0} \frac{f'(x)-f'(0)}{x}$. For $x \neq 0$, the derivative is $f'(x) = \frac{x\cos x - \sin x}{x^2}$.

$$f''(0) = \lim_{x\to 0} \frac{\frac{x\cos x-\sin x}{x^2} - 0}{x} = \lim_{x\to 0} \frac{x\cos x - \sin x}{x^3}$$

Applying L'Hopital's Rule:

$$= \lim_{x\to 0} \frac{-x\sin x}{3x^2} = -\frac{1}{3}\lim_{x\to 0} \frac{\sin x}{x} = \boxed{-\frac{1}{3}}$$

2. Tangent Addition Formula (MVT Proof)

Fix y and define the function:

$$g(x) = \arctan\left(\frac{\tan x + \tan y}{1 - \tan x \tan y}\right) - x$$

We calculate $g'(x)$. Let $u = \frac{\tan x + \tan y}{1 - \tan x \tan y}$. Then $g'(x) = \frac{u'}{1+u^2} - 1$. After applying the quotient rule and simplifying the algebra, we find that $\frac{u'}{1+u^2} = 1$.

$$g'(x) = 1 - 1 = 0$$

Since $g'(x) = 0$ for all x, $g(x)$ is a constant C (by the Corollary of MVT). Evaluate at $x = 0$:

$$C = g(0) = \arctan(\tan y) - 0 = y$$

Thus $\arctan(\dots) = x + y$. Taking the tangent of both sides proves the identity.

3. Orthogonality Integral

Let $f(x) = \sin(mx)\cos(nx)$.

- $\sin(mx)$ is an **odd** function ($f(-x) = -f(x)$).
- $\cos(nx)$ is an **even** function ($f(-x) = f(x)$).

The product is an **odd** function. The integral of an odd function over the symmetric interval $[-\pi, \pi]$ is $\boxed{0}$.

4. L'Hopital's Rule

(i) Form $\frac{0}{0}$. Apply L'Hopital's Rule twice:

$$\lim_{x\to 0} \frac{e^x - 1 - x - x^2}{x^3} \to \lim_{x\to 0} \frac{e^x - 1 - 2x}{3x^2} \to \lim_{x\to 0} \frac{e^x - 2}{6x}$$

The numerator approaches -1 and the denominator 0. The limit **diverges**.

(ii) Form 1^∞. Let $L = \lim(\tan x)^{\tan 2x}$.

$$\ln L = \lim_{x\to\pi/4} \frac{\ln(\tan x)}{\cot 2x} \quad \left(\text{Form } \frac{0}{0}\right)$$

$$= \lim_{x\to\pi/4} \frac{\sec^2 x/\tan x}{-2\csc^2(2x)} = \frac{2/1}{-2(1)} = -1$$

Thus $\ln L = -1 \implies \boxed{L = e^{-1}}$.

5. Definition of e

Let $y = (1 + 1/x)^x$. Then $\ln y = x \ln(1 + 1/x)$. Let $h = 1/x$.

$$\lim_{x \to \infty} \ln y = \lim_{h \to 0} \frac{\ln(1 + h)}{h}$$

This limit represents the derivative of $\ln(t)$ at $t = 1$, which is $1/1 = 1$. Since $\ln y \to 1$, we have $y \to \boxed{e}$.

6. Integral Equation (FTC)

Given $f(x) = \int_0^x f(t)dt$. By the Fundamental Theorem of Calculus, $f'(x) = f(x)$. This differential equation $y' = y$ has the general solution $f(x) = Ce^x$. Using the initial condition at $x = 0$:

$$f(0) = \int_0^0 f(t)dt = 0 \implies Ce^0 = 0 \implies C = 0$$

Therefore, $\boxed{f(x) = 0}$.