MATH 145: Calculus for Engineering and Science I
Recitation 9
Solution Key

December 22th, 2025

1. Evaluate the following integrals (integration by parts):

(i) $\displaystyle\int x^2 \sin x \, dx$

(ii) $\displaystyle\int \sec^3 x \, dx$

(iii) $\displaystyle\int \cos(\log x) \, dx$

Solutions:

i. **Evaluate** $\displaystyle\int x^2 \sin x \, dx$

Solution: We use integration by parts. Let $u = x^2$ and $dv = \sin x \, dx$. Then $du = 2x \, dx$ and $v = -\cos x$.

$$I = -x^2 \cos x - \int (-\cos x)(2x) \, dx = -x^2 \cos x + 2 \int x \cos x \, dx$$

We apply integration by parts again for $\int x \cos x \, dx$. Let $u = x$ and $dv = \cos x \, dx$. Then $du = dx$ and $v = \sin x$.

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x$$

Substituting this back into the original expression:

$$\boxed{I = -x^2 \cos x + 2x \sin x + 2 \cos x + C}$$

ii. **Evaluate** $\displaystyle\int \sec^3 x \, dx$

Solution: Split the integrand: $\sec^3 x = \sec x \cdot \sec^2 x$. Let $u = \sec x$ and $dv = \sec^2 x \, dx$. Then $du = \sec x \tan x \, dx$ and $v = \tan x$.

$$I = \sec x \tan x - \int \tan x (\sec x \tan x) \, dx = \sec x \tan x - \int (\sec^2 x - 1) \sec x \, dx$$

$$I = \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$$

$$I = \sec x \tan x - I + \ln|\sec x + \tan x|$$

Solving for I:

$$2I = \sec x \tan x + \ln|\sec x + \tan x|$$

$$\boxed{I = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C}$$

iii. **Evaluate** $\int \cos(\log x)\, dx$

Solution: Let $t = \log x \implies x = e^t$ and $dx = e^t dt$. The integral becomes $\int e^t \cos t\, dt$. Using integration by parts twice for $I = \int e^t \cos t\, dt$:

(a) $u = \cos t, dv = e^t dt \implies I = e^t \cos t + \int e^t \sin t\, dt$.

(b) For $\int e^t \sin t\, dt$: $u = \sin t, dv = e^t dt \implies \int e^t \sin t\, dt = e^t \sin t - I$.

Combining these: $I = e^t \cos t + e^t \sin t - I \implies 2I = e^t(\cos t + \sin t)$. Substituting $e^t = x$ and $t = \log x$:

$$\boxed{I = \frac{x}{2}(\cos(\log x) + \sin(\log x)) + C}$$

2. Evaluate the following integrals (substitution):

(i) $\displaystyle\int \frac{dx}{x\sqrt{x^2-1}}$

(ii) $\displaystyle\int x^3\sqrt{1-x^2}\,dx$

(iii) $\displaystyle\int \sqrt{x^2+1}\,dx$

Solutions:

i. **Evaluate** $\displaystyle\int \frac{dx}{x\sqrt{x^2-1}}$

Solution: Let $x = \sec\theta$, so $dx = \sec\theta\tan\theta\,d\theta$.

$$\int \frac{\sec\theta\tan\theta\,d\theta}{\sec\theta\sqrt{\sec^2\theta-1}} = \int \frac{\sec\theta\tan\theta}{\sec\theta\tan\theta}\,d\theta = \int d\theta = \theta + C$$

Since $x = \sec\theta \implies \theta = \sec^{-1}x$:

$$\boxed{I = \sec^{-1}x + C}$$

ii. **Evaluate** $\displaystyle\int x^3\sqrt{1-x^2}\,dx$

Solution: Let $u = 1-x^2 \implies du = -2x\,dx$ and $x^2 = 1-u$.

$$\int x^2\sqrt{1-x^2}(x\,dx) = \int (1-u)\sqrt{u}\left(-\frac{1}{2}du\right) = -\frac{1}{2}\int (u^{1/2}-u^{3/2})\,du$$

$$= -\frac{1}{2}\left(\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2}\right) + C = -\frac{1}{3}u^{3/2} + \frac{1}{5}u^{5/2} + C$$

$$\boxed{I = -\frac{1}{3}(1-x^2)^{3/2} + \frac{1}{5}(1-x^2)^{5/2} + C}$$

iii. **Evaluate** $\displaystyle\int \sqrt{x^2+1}\,dx$

Solution: Let $x = \tan\theta \implies dx = \sec^2\theta\,d\theta$.

$$\int \sqrt{\tan^2\theta+1}\,\sec^2\theta\,d\theta = \int \sec^3\theta\,d\theta$$

From Problem 1(ii), we know the integral of $\sec^3\theta$:

$$\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta+\tan\theta| + C$$

Substituting $\tan\theta = x$ and $\sec\theta = \sqrt{x^2+1}$:

$$\boxed{I = \frac{1}{2}x\sqrt{x^2+1} + \frac{1}{2}\ln\left|x+\sqrt{x^2+1}\right| + C}$$

3. Evaluate the following integrals (mixture of methods):

(i) $\displaystyle\int \frac{x}{1+\sin x}\,dx$

(ii) $\displaystyle\int \log\sqrt{1+x^2}\,dx$

(iii) $\displaystyle\int \frac{1}{x^6+1}\,dx$

Solutions:

i. **Evaluate** $\displaystyle\int \frac{x}{1+\sin x}\,dx$

Solution: Multiply by the conjugate $(1-\sin x)/(1-\sin x)$:

$$\int \frac{x(1-\sin x)}{\cos^2 x}\,dx = \int x\sec^2 x\,dx - \int x\sec x\tan x\,dx$$

Using integration by parts on both terms (differentiating x):

$$\int x\sec^2 x\,dx = x\tan x - \ln|\sec x|$$

$$\int x\sec x\tan x\,dx = x\sec x - \ln|\sec x + \tan x|$$

Subtracting the second from the first:

$$\boxed{I = x(\tan x - \sec x) + \ln|1+\sin x| + C}$$

ii. **Evaluate** $\displaystyle\int \log\sqrt{1+x^2}\,dx$

Solution: Rewrite as $\frac{1}{2}\int \log(1+x^2)\,dx$. Use integration by parts with $u = \log(1+x^2)$, $dv = dx$.

$$I = \frac{1}{2}\left[x\log(1+x^2) - \int \frac{2x^2}{1+x^2}\,dx\right] = \frac{1}{2}x\log(1+x^2) - \int\left(1 - \frac{1}{1+x^2}\right)dx$$

$$\boxed{I = x\log\sqrt{1+x^2} - x + \arctan x + C}$$

iii. **Evaluate** $\displaystyle\int \frac{1}{x^6+1}\,dx$

Solution: Factor the denominator: $x^6+1 = (x^2+1)(x^4-x^2+1) = (x^2+1)(x^2-\sqrt{3}x+1)(x^2+\sqrt{3}x+1)$. Using partial fractions:

$$\frac{1}{x^6+1} = \frac{A}{x^2+1} + \frac{Bx+C}{x^2-\sqrt{3}x+1} + \frac{Dx+E}{x^2+\sqrt{3}x+1}$$

Solving for coefficients yields the integral:

$$\boxed{\frac{\arctan x}{3} + \frac{\arctan(2x+\sqrt{3}) + \arctan(2x-\sqrt{3})}{6} + \frac{\sqrt{3}}{12}\ln\left(\frac{x^2+\sqrt{3}x+1}{x^2-\sqrt{3}x+1}\right) + C}$$