MATH 145 Recitation 3 - Solution Key

1. Find the following limits

1.1. $\lim_{x\to 1} \frac{x^2-1}{x+1}$

Solution. *The function* $f(x) = \frac{x^2-1}{x+1}$ *is continuous at* $x = 1$ *because the denominator is* $1 + 1 = 2 \neq 0$. *We can find the limit by* ***direct substitution:***

$$\lim_{x\to 1} \frac{x^2-1}{x+1} = \frac{(1)^2-1}{1+1} = \frac{1-1}{2} = \frac{0}{2} = 0$$

1.2. $\lim_{x\to 3} \frac{x^3-8}{x-2}$

Solution. *The function* $f(x) = \frac{x^3-8}{x-2}$ *is continuous at* $x = 3$ *because the denominator is* $3 - 2 = 1 \neq 0$. *We can find the limit by* ***direct substitution:***

$$\lim_{x\to 3} \frac{x^3-8}{x-2} = \frac{(3)^3-8}{3-2} = \frac{27-8}{1} = \frac{19}{1} = 19$$

1.3. $\lim_{h\to 0} \frac{\sqrt{a+h}-\sqrt{a}}{h}$

Solution. *This limit is in the* $\frac{0}{0}$ *indeterminate form (assuming* $a \geq 0$*). We* ***rationalize the numerator*** *by multiplying by its conjugate. We assume* $a > 0$.

$$\begin{aligned}
\lim_{h\to 0} \frac{\sqrt{a+h}-\sqrt{a}}{h} &= \lim_{h\to 0} \frac{\sqrt{a+h}-\sqrt{a}}{h} \cdot \left(\frac{\sqrt{a+h}+\sqrt{a}}{\sqrt{a+h}+\sqrt{a}}\right) \\
&= \lim_{h\to 0} \frac{(\sqrt{a+h})^2-(\sqrt{a})^2}{h(\sqrt{a+h}+\sqrt{a})} \\
&= \lim_{h\to 0} \frac{(a+h)-a}{h(\sqrt{a+h}+\sqrt{a})} \\
&= \lim_{h\to 0} \frac{h}{h(\sqrt{a+h}+\sqrt{a})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{a+h}+\sqrt{a}} \quad (\textit{Since } h \neq 0) \\
&= \frac{1}{\sqrt{a+0}+\sqrt{a}} = \frac{1}{\sqrt{a}+\sqrt{a}} = \frac{1}{2\sqrt{a}}
\end{aligned}$$

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2. $\epsilon - \delta$ Proofs of Limits

2.1. $f(x) = x[3 - \cos(x^2)]$, $a = 0$

Solution. ***Step 1: Find the limit*** l. $l = \lim_{x\to 0} x[3 - \cos(x^2)] = 0 \cdot [3 - \cos(0)] = 0 \cdot (3 - 1) = 0$. *We must prove* $\lim_{x\to 0} f(x) = 0$.

Step 2: $\epsilon - \delta$ ***Proof.*** *We need to show that for any* $\epsilon > 0$, *there exists a* $\delta > 0$ *such that if* $0 < |x - 0| < \delta$, *then* $|f(x) - 0| < \epsilon$.

We want $|x[3 - \cos(x^2)] - 0| < \epsilon$, *which is* $|x| \cdot |3 - \cos(x^2)| < \epsilon$. *We know* $-1 \leq \cos(x^2) \leq 1$. *This implies* $1 \geq -\cos(x^2) \geq -1$. *Adding 3, we get* $4 \geq 3 - \cos(x^2) \geq 2$. *Thus, the term is always positive,* *and $|3-\cos(x^2)| \leq 4$. Our inequality becomes $|x| \cdot |3-\cos(x^2)| \leq 4|x|$. We want $4|x| < \epsilon$, which means $|x| < \epsilon/4$. This suggests $\delta = \epsilon/4$.*

Formal Proof. *Let $\epsilon > 0$ be given. Choose $\delta = \epsilon/4$. If $0 < |x-0| < \delta$, then $|x| < \epsilon/4$. Consider $|f(x)-0| = |x[3-\cos(x^2)]| = |x| \cdot |3-\cos(x^2)|$. Since $|3-\cos(x^2)| \leq 4$ for all x, we have*

$$|f(x)-0| \leq |x| \cdot 4$$

Since $|x| < \epsilon/4$, we substitute

$$|f(x)-0| < (\epsilon/4) \cdot 4 = \epsilon$$

Thus, $0 < |x| < \delta$ implies $|f(x)-0| < \epsilon$.

2.2. $f(x) = x^2 + 5x - 2$**,** $a = 2$

Solution. ***Step 1: Find the limit*** l. *$l = \lim_{x\to 2}(x^2+5x-2) = (2)^2 + 5(2) - 2 = 4 + 10 - 2 = 12$. We must prove $\lim_{x\to 2} f(x) = 12$.*

Step 2: $\epsilon - \delta$ ***Proof.*** *We need to show that for any $\epsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x-2| < \delta$, then $|(x^2+5x-2) - 12| < \epsilon$.*

We want $|x^2+5x-14| < \epsilon$, which factors to $|(x-2)(x+7)| < \epsilon$. This is $|x-2|\cdot|x+7| < \epsilon$. We need to bound $|x+7|$ for x near $a = 2$. Let's restrict $\delta \leq 1$. Then $|x-2| < 1 \implies -1 < x-2 < 1 \implies 1 < x < 3$. If $1 < x < 3$, then $8 < x+7 < 10$. So, $|x+7| < 10$. Our inequality becomes $|x-2|\cdot|x+7| < |x-2|\cdot 10$. We want $|x-2| \cdot 10 < \epsilon$, so $|x-2| < \epsilon/10$. We need both $\delta \leq 1$ and $\delta \leq \epsilon/10$. We choose $\delta = \min(1, \epsilon/10)$.

Formal Proof: *Let $\epsilon > 0$ be given. Choose $\delta = \min(1, \epsilon/10)$. If $0 < |x-2| < \delta$, then:*

1. *$|x-2| < 1$, which implies $1 < x < 3$, and therefore $|x+7| < 10$.*
2. *$|x-2| < \epsilon/10$.*

Consider $|f(x)-12| = |x^2+5x-14| = |(x-2)(x+7)| = |x-2| \cdot |x+7|$. Using our two conditions:

$$|f(x)-12| < (\epsilon/10) \cdot 10 = \epsilon$$

Thus, $0 < |x-2| < \delta$ implies $|f(x)-12| < \epsilon$.

2.3. $f(x) = \frac{x}{2-\sin^2 x}$**,** $a = 0$

Solution. ***Step 1: Find the limit*** l. *$l = \lim_{x\to 0} \frac{x}{2-\sin^2 x} = \frac{0}{2-\sin^2(0)} = \frac{0}{2-0} = 0$. We must prove $\lim_{x\to 0} f(x) = 0$.*

Step 2: Formal $\epsilon - \delta$ ***Proof.*** *We need to show that for any $\epsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x-0| < \delta$, then $|\frac{x}{2-\sin^2 x} - 0| < \epsilon$.*

We want $\frac{|x|}{|2-\sin^2 x|} < \epsilon$. We need to bound the denominator. We know $0 \leq \sin^2(x) \leq 1$. Multiplying by -1: $0 \geq -\sin^2(x) \geq -1$. Adding 2: $2 \geq 2-\sin^2(x) \geq 1$. Since the denominator is always ≥ 1, we have $|2-\sin^2(x)| \geq 1$. Therefore, $\frac{|x|}{|2-\sin^2 x|} \leq \frac{|x|}{1} = |x|$. We want $|x| < \epsilon$. This suggests $\delta = \epsilon$.

Formal Proof: *Let $\epsilon > 0$ be given. Choose $\delta = \epsilon$. If $0 < |x-0| < \delta$, then $|x| < \epsilon$. Consider $|f(x)-0| = |\frac{x}{2-\sin^2 x}| = \frac{|x|}{|2-\sin^2 x|}$. As shown above, $|2-\sin^2 x| \geq 1$, so $\frac{1}{|2-\sin^2 x|} \leq 1$.*

$$|f(x)-0| \leq |x| \cdot 1 = |x|$$

Since $|x| < \epsilon$, we have:

$$|f(x)-0| < \epsilon$$

Thus, $0 < |x| < \delta$ implies $|f(x)-0| < \epsilon$.

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3. Find the following limits in terms of $\alpha = \lim_{x \to 0} \frac{\sin x}{x}$

3.1. $\lim_{x \to \infty} \frac{\sin x}{x}$

Solution. *We use the Squeeze Theorem. We know* $-1 \le \sin(x) \le 1$ *for all* x. *As* $x \to \infty$, *we can assume* $x > 0$. *Dividing by* x, *we get*

$$\frac{-1}{x} \le \frac{\sin(x)}{x} \le \frac{1}{x}$$

We take the limit of the outer functions

$$\lim_{x \to \infty} \frac{-1}{x} = 0 \quad \textit{and} \quad \lim_{x \to \infty} \frac{1}{x} = 0$$

By the Squeeze Theorem, the limit of the middle function must also be 0.

$$\lim_{x \to \infty} \frac{\sin x}{x} = 0$$

3.2. $\lim_{x \to \infty} x \sin \frac{1}{x}$

Solution. *We use a substitution. Let* $t = \frac{1}{x}$. *As* $x \to \infty$, *we have* $t \to 0$. *The limit becomes*

$$\lim_{x \to \infty} x \sin \frac{1}{x} = \lim_{t \to 0} \frac{1}{t} \sin(t) = \lim_{t \to 0} \frac{\sin(t)}{t}$$

This is the definition of α.

$$\lim_{x \to \infty} x \sin \frac{1}{x} = \alpha$$

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4. Prove that $\lim_{x \to a} f(x) = \lim_{h \to 0} f(a + h)$

Proof. ($\Rightarrow$) Assume $\lim_{x \to a} f(x) = L$. This means that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$. We want to show $\lim_{h \to 0} f(a + h) = L$. Let $\epsilon > 0$ be given. Use the same δ from the assumption. Assume $0 < |h - 0| < \delta$, which is $0 < |h| < \delta$. Let $x = a + h$. Then $h = x - a$. Substituting $h = x - a$ into our assumption $0 < |h| < \delta$ gives $0 < |x - a| < \delta$. By our initial assumption, this implies $|f(x) - L| < \epsilon$. Substituting $x = a + h$ back into this conclusion gives $|f(a+h) - L| < \epsilon$. Thus, for any $\epsilon > 0$, we found a $\delta > 0$ such that $0 < |h| < \delta$ implies $|f(a+h) - L| < \epsilon$.

($\Leftarrow$) Assume $\lim_{h \to 0} f(a + h) = L$. This means that for every $\epsilon > 0$, there exists a $\delta > 0$ such that if $0 < |h| < \delta$, then $|f(a + h) - L| < \epsilon$. We want to show $\lim_{x \to a} f(x) = L$. Let $\epsilon > 0$ be given. Use the same δ from the assumption. Assume $0 < |x - a| < \delta$. Let $h = x - a$. Then $x = a + h$. Substituting $h = x - a$ into our assumption $0 < |x - a| < \delta$ gives $0 < |h| < \delta$. By our initial assumption, this implies $|f(a + h) - L| < \epsilon$. Substituting $a + h = x$ back into this conclusion gives $|f(x) - L| < \epsilon$. Thus, for any $\epsilon > 0$, we found a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$.

Since both statements imply each other, they are equivalent. ■

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5. For which of the following functions f is there a continuous function F with domain $\mathbb{R}$ such that $F(x) = f(x)$ for all x in the domain of f?

Solution. *This is possible if and only if all discontinuities of* $f(x)$ *are* ***removable discontinuities****.* *This occurs at a point* $x = c$ *if* $f(c)$ *is undefined but* $\lim_{x \to c} f(x)$ *exists.*

5.1. $f(x) = \frac{x^2 - 4}{x - 2}$

The domain of $f(x)$ *is* $\mathbb{R} \setminus \{2\}$. *We check the limit at* $x = 2$.

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \to 2} (x + 2) = 4$$

Since the limit exists, the discontinuity is removable. ***Yes, this function can be extended.*** *The continuous extension is* $F(x) = x + 2$.

5.2. $f(x) = \frac{|x|}{x}$

The domain of $f(x)$ *is* $\mathbb{R} \setminus \{0\}$. *We check the limit at* $x = 0$ *by checking the one-sided limits.*

$$\lim_{x \to 0^+} \frac{|x|}{x} = \lim_{x \to 0^+} \frac{x}{x} = \lim_{x \to 0^+} 1 = 1$$

$$\lim_{x \to 0^-} \frac{|x|}{x} = \lim_{x \to 0^-} \frac{-x}{x} = \lim_{x \to 0^-} -1 = -1$$

Since the right-hand limit (1) does not equal the left-hand limit (-1), the overall limit $\lim_{x \to 0} f(x)$ ***does not exist****. This is a non-removable (jump) discontinuity.* ***No, this function cannot be extended*** *to be continuous on all of* $\mathbb{R}$.

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6. Suppose that f is continuous at a and $f(a) = 0$. Prove that if $\alpha \neq 0$, then $f + \alpha$ is nonzero in some open interval containing a.

Solution. *Proof.* Let $g(x) = f(x) + \alpha$. We are given that $f(x)$ is continuous at a and $f(a) = 0$. Since $f(x)$ and the constant function $h(x) = \alpha$ are continuous at a, their sum $g(x) = f(x) + \alpha$ is also continuous at a. The value of $g(x)$ at a is $g(a) = f(a) + \alpha = 0 + \alpha = \alpha$.

By the definition of continuity for $g(x)$ at a, for any $\epsilon > 0$, there exists a $\delta > 0$ such that if $|x - a| < \delta$, then $|g(x) - g(a)| < \epsilon$.

We are given $\alpha \neq 0$, so $|\alpha| > 0$. Let us choose $\epsilon = \frac{|\alpha|}{2}$. Since $|\alpha| > 0$, this is a valid positive ϵ. For this ϵ, there must exist some $\delta > 0$ such that if $|x - a| < \delta$, then $|g(x) - g(a)| < \frac{|\alpha|}{2}$.

Substitute $g(a) = \alpha$:

$$|g(x) - \alpha| < \frac{|\alpha|}{2}$$

Now, we use the reverse triangle inequality ($||A| - |B|| \leq |A - B|$), which implies $|A - B| \geq |A| - |B|$. Let $A = \alpha$ and $B = g(x)$. Then $|g(x) - \alpha| \geq |\alpha| - |g(x)|$. Combining this with our inequality, we obtain

$$|\alpha| - |g(x)| \leq |g(x) - \alpha| < \frac{|\alpha|}{2} \implies \frac{|\alpha|}{2} < |g(x)|.$$

Since $\frac{|\alpha|}{2} > 0$, this proves that $|g(x)| > 0$, which means $g(x) \neq 0$. This holds for all x satisfying $|x - a| < \delta$, which is the open interval $(a - \delta, a + \delta)$. Thus, $g(x) = f(x) + \alpha$ is nonzero in the open interval $(a - \delta, a + \delta)$ containing a. ∎